Discrete Chebyshev nets and a universal permutability theorem

2007 J. Phys. A: Math. Theor. 40 4775

(http://iopscience.iop.org/1751-8121/40/18/007)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 4775–4801 doi:10.1088/1751-8113/40/18/007

Discrete Chebyshev nets and a universal permutability theorem

W K Schief

Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany

Received 31 January 2007, in final form 21 March 2007
Published 17 April 2007
Online at stacks.iop.org/JPhysA/40/4775

Abstract

The Pohlmeyer–Lund–Regge system which was set down independently in the contexts of Lagrangian field theories and the relativistic motion of a string and which played a key role in the development of a geometric interpretation of soliton theory is known to appear in a variety of important guises such as the vectorial Lund–Regge equation, the $O(4)$ nonlinear σ-model and the $SU(2)$ chiral model. Here, it is demonstrated that these avatars may be discretized in such a manner that both integrability and equivalence are preserved. The corresponding discretization procedure is geometric and algebraic in nature and based on discrete Chebyshev nets and generalized discrete Lelieuvre formulae. In connection with the derivation of associated Bäcklund transformations, it is shown that a generalized discrete Lund–Regge equation may be interpreted as a universal permutability theorem for integrable equations which admit commuting matrix Darboux transformations acting on $su(2)$ linear representations. Three-dimensional coordinate systems and lattices of 'Lund–Regge' type related to particular continuous and discrete Zakharov–Manakov systems are obtained as a by-product of this analysis.

PACS numbers: 02.30.Ik, 02.30.Jr, 02.40.Hw, 05.45.Yv

1. Introduction

The discovery by Lund and Regge [1] in 1976 that the AKNS scattering problem [2] for the integrable sine-Gordon equation and its associated 'time evolution' constitute nothing but an $su(2)$-valued parameter-dependent generalization of the Gauß–Weingarten equations for classical pseudospherical surfaces marked the beginning of an extensive geometric treatment of modern soliton theory [3]. The system of nonlinear partial differential equations set down by Lund and Regge in connection with the relativistic motion of a string in a uniform and static external field was independently proven to be an integrable generalization of the sine-Gordon equation by Pohlmeyer [4] who was concerned with O_n-invariant Lagrangian field theories.

Lund [5] then showed that what is now known as the Pohlmeyer–Lund–Regge system may also be interpreted as the Gauß–Mainardi–Codazzi equations for particular surfaces in S^3. This work was set in a broader context by Sym [6] who demonstrated that the connection between soliton theory and classical differential geometry is natural and profound. However, it turns out that differential geometers such as Bäcklund, Bianchi and Darboux may, in fact, be regarded as the classical pioneers in this area (see [7] and references therein).

The important observation by Bobenko and Pinkall [8] in 1996 that the discrete analogues of pseudospherical surfaces which were proposed independently by Sauer [9] and Wunderlich [10] in 1950 and 1951 respectively are governed by Hirota's integrable discretization of the sine-Gordon equation [11] demonstrated that the above-mentioned geometric connection extends to discrete integrable systems. In fact, it has become evident that the link between integrable differential geometry and its discrete analogue is provided by fundamental transformations known as Bäcklund transformations which were studied in great detail by differential geometers around the turn of the 19th century [7]. In the past decade, the subject of 'discrete differential geometry', which is concerned with the development of discrete models of differential-geometric objects which preserve some of the fundamental properties of their continuous counterparts such as integrability, has become a field of extensive research (see [12] and references therein). The key role of Bäcklund transformations and their associated permutability theorems [7] in both the continuous and discrete settings has thereby been revealed.

Against the background of the developments in the past three decades, we here return to the pioneering work of Lund and Regge and investigate in detail its significance in the modern context of discrete differential geometry. As a by-product, novel geometric and algebraic aspects of the original (continuous) Pohlmeyer–Lund–Regge system are also uncovered. The aim of this investigation is to establish a complete correspondence between the discrete and continuous settings and to demonstrate how a universal permutability theorem naturally arises in this endeavour.

In view of the discrete setting, we begin with a review of some known algebraic and geometric properties of the Lund–Regge equation which represents a vectorial avatar of the Pohlmeyer–Lund–Regge system and its relation to the $O(4)$ nonlinear σ-model and the $SU(2)$ chiral model. We recall the definition of Chebyshev nets on surfaces and present a novel generalization of the classical Lelieuvre formulae for pseudospherical surfaces which provide a natural link between the $O(4)$ nonlinear σ-model and pairs of Lund–Regge surfaces. It thereby emerges that the Chebyshev net parametrization of these pairs of Lund–Regge surfaces corresponds to a parametrization of the associated 'mid-surfaces' in terms of asymptotic coordinates. Section 2 is formulated in such a manner that, remarkably, it may be directly 'translated' into the discrete language. Thus, based on the standard concept of discrete Chebyshev nets, discrete Lund–Regge surfaces are introduced and their connection with discrete versions of the $O(4)$ nonlinear σ-model and the $SU(2)$ chiral model is discussed. In particular, generalized discrete Lelieuvre formulae are presented and shown to encode pairs of discrete Lund–Regge surfaces and corresponding mid-surfaces which, once again, constitute standard discrete asymptotic lattices.

The discrete Lund–Regge equation is naturally embedded in a class of equations which depends on two arbitrary functions of one variable. In section 4, it is shown that there exists a Bäcklund transformation which acts within this class of 'generalized Lund–Regge equations' and which, in fact, preserves any particular choice of these two functions. We then exploit this result to prove that it is consistent to demand that the generalized discrete Lund–Regge equation regarded as a four-point relation holds on any face of a $\mathbb{Z}^3$ lattice. In the natural continuum limit, this implies that there exist coordinate systems in $\mathbb{R}^3$ for which the coordinate surfaces are of generalized Lund–Regge type. In particular, (standard) Lund–Regge surfaces may be used to construct special three-dimensional coordinate systems.

2. Chebyshev nets and the integrable Pohlmeyer–Lund–Regge system

In this section, we give a brief summary of various connections between the Pohlmeyer–Lund–Regge system, the $O(4)$ nonlinear σ-model and the $SU(2)$ chiral model. In addition, we extend the link between pseudospherical surfaces and their spherical representation provided by the classical Lelieuvre formulae to the case of pairs of Lund–Regge surfaces and the $O(4)$ nonlinear σ-model.

2.1. Chebyshev nets

It is well known [13] that any surface Σ in a three-dimensional Euclidean space with position vector $\boldsymbol{r}$ may be locally parametrized in such a way that the first fundamental form is given by

$$\mathrm{I} = \mathrm{d}\boldsymbol{r}^2 = \mathrm{d}x^2 + 2\cos 2\theta\, \mathrm{d}x\, \mathrm{d}y + \mathrm{d}y^2, \tag{2.1}$$

where 2θ constitutes the angle between the coordinate lines $x = \text{const}$ and $y = \text{const}$. The latter two families form a net on Σ which commonly bears the name of Chebyshev who was the first to undertake a detailed investigation of such nets in connection with 'the cutting of clothes' [14]. A Chebyschev net of curves on a surface may be characterized by the requirement that the lengths of opposite sides of any (curved) quadrilateral formed by two pairs of curves be the same. It is evident that the above fundamental form is obtained by demanding that the coordinate lines be parametrized in terms of arc length. In general, the conditions for a coordinate system to define a Chebyshev net are given by

$$\boldsymbol{r}_x^2 = f(x), \qquad \boldsymbol{r}_y^2 = g(y). \tag{2.2}$$

Suitable reparametrizations of the coordinate lines then lead to the arc length constraints $\boldsymbol{r}_x^2 = \boldsymbol{r}_y^2 = 1$. Differentiation of (2.2) yields

$$\boldsymbol{r}_x \cdot \boldsymbol{r}_{xy} = 0, \qquad \boldsymbol{r}_y \cdot \boldsymbol{r}_{xy} = 0 \tag{2.3}$$

so that a surface Σ is parametrized in terms of 'Chebyshev coordinates' if and only if the position vector $\boldsymbol{r}(x, y)$ obeys a differential equation of the form

$$\boldsymbol{r}_{xy} = \sigma \boldsymbol{r}_x \times \boldsymbol{r}_y, \qquad \sigma = \sigma(x, y). \tag{2.4}$$

This is equivalent to stating that

$$\boldsymbol{r}_{xy} \parallel \hat{\boldsymbol{N}}, \tag{2.5}$$

where $\hat{\boldsymbol{N}} = \boldsymbol{r}_x \times \boldsymbol{r}_y / |\boldsymbol{r}_x \times \boldsymbol{r}_y|$ denotes the unit normal to Σ.

2.2. The Pohlmeyer–Lund–Regge system

Any prescribed scalar function σ in the differential equation (2.4) corresponds to a particular class of surfaces bearing the Chebyshev nets. The simplest choice $\sigma = 1$ leads to

$$\boldsymbol{r}_{xy} = \boldsymbol{r}_x \times \boldsymbol{r}_y \tag{2.6}$$

which has been demonstrated by Lund and Regge [1] to represent the relativistic motion of a string in a uniform and static external field. Based on the canonical metric (2.1), it may be shown that the second fundamental form of the surface $\Sigma : \boldsymbol{r} = \boldsymbol{r}(x, y)$ admits the parametrization

$$\mathrm{II} = -\mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\hat{\boldsymbol{N}} = 2\cot\theta\varphi_x\, \mathrm{d}x^2 + 2\sin 2\theta\, \mathrm{d}x\, \mathrm{d}y + 2\cot\theta\varphi_y\, \mathrm{d}y^2. \tag{2.7}$$

The associated Gauß–Mainardi–Codazzi equations [13], which completely encode the Lund–Regge equation (2.6), then read

$$\begin{aligned}\theta_{xy} + \frac{\cos\theta}{\sin^3\theta}\varphi_x\varphi_y &= \sin\theta\cos\theta \\ (\cot^2\theta\varphi_x)_y &= (\cot^2\theta\varphi_y)_x.\end{aligned} \tag{2.8}$$

The above system is invariant under $\varphi \to -\varphi$. Since the second fundamental form (2.7) is not preserved by this invariance, it is evident that any Lund–Regge surface Σ naturally admits another Lund–Regge surface $\tilde{\Sigma}$ which is uniquely defined up to rigid motions. The two surfaces coincide if $\varphi = 0$, in which case (2.8) reduces to the sine-Gordon equation

$$2\theta_{xy} = \sin 2\theta \tag{2.9}$$

and the surface $\Sigma = \tilde{\Sigma}$ constitutes a pseudospherical surface [15] since the Gaußian curvature is given by

$$\mathcal{K} = \frac{\det \mathrm{II}}{\det \mathrm{I}} = -1. \tag{2.10}$$

The classical Lelieuvre formulae [15]

$$\boldsymbol{r}_x = \hat{\boldsymbol{N}} \times \hat{\boldsymbol{N}}_x, \qquad \boldsymbol{r}_y = \hat{\boldsymbol{N}}_y \times \hat{\boldsymbol{N}}, \tag{2.11}$$

which provide the link between a pseudospherical surface and its spherical representation, then imply, on elimination of $\boldsymbol{r}$, that $\hat{\boldsymbol{N}}_{xy} \parallel \hat{\boldsymbol{N}}$ and hence $\hat{\boldsymbol{N}}$ is governed by the nonlinear σ-model

$$\hat{\boldsymbol{N}}_{xy} + (\hat{\boldsymbol{N}}_x \cdot \hat{\boldsymbol{N}}_y)\hat{\boldsymbol{N}} = \boldsymbol{0}, \qquad \hat{\boldsymbol{N}}^2 = 1. \tag{2.12}$$

In fact, Pohlmeyer [4] has shown that the unconstrained nonlinear system (2.8) is equivalent to the $O(4)$ nonlinear σ-model

$$\mathsf{N}_{xy} + (\mathsf{N}_x \cdot \mathsf{N}_y)\mathsf{N} = 0, \qquad \mathsf{N} \in S^3 \tag{2.13}$$

and admits a linear representation in the sense of soliton theory [3]. Moreover, Lund [5] has observed that the integrable Pohlmeyer–Lund–Regge system is nothing but the Gauß–Mainardi–Codazzi equations associated with the class of surfaces $\Sigma \subset S^3$ defined by (2.13).

2.3. Chiral and nonlinear σ-models

At the surface level, the connection between the Pohlmeyer–Lund–Regge system and the $O(4)$ nonlinear σ-model is established in the following manner [16]. We first identify the space of quaternions $\mathbb{H}$ with a four-dimensional Euclidean space $\mathbb{R}^4$ via

$$\mathbb{R}^4 \ni (a, b, c, d) \quad \leftrightarrow \quad (a\mathbb{1} + b\mathbb{i} + c\mathbb{j} + d\mathbb{k}) \in \mathbb{H}, \tag{2.14}$$

where the matrices $\mathbb{1}, \mathbb{i}, \mathbb{j}, \mathbb{k}$ are defined by

$$\mathbb{1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \mathbb{i} = \begin{pmatrix}0 & -\mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}, \qquad \mathbb{j} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad \mathbb{k} = \begin{pmatrix}-\mathrm{i} & 0\\ 0 & \mathrm{i}\end{pmatrix}. \tag{2.15}$$

It is noted in passing that $\mathbb{H}$ admits the two characterizations

$$\begin{aligned}\mathbb{H} &= \left\{H \in \mathbb{C}^{2,2} : M\bar{H} = HM, M = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\} \\ \mathbb{H} &= \{H \in \mathbb{C}^{2,2} : H^\dagger H \sim \mathbb{1}, H^\dagger + H \sim \mathbb{1}\}\end{aligned} \tag{2.16}$$

so that

$$\mathbb{H} \subset \mathfrak{G} = \{\Phi \in \mathbb{C}^{2,2} : \Phi^\dagger\Phi \sim \mathbb{1}\}. \tag{2.17}$$

In particular, we adopt the isomorphism of the three-dimensional subspaces $\mathbb{R}^4 \supset \mathbb{R}^3 \cong su(2) \subset \mathbb{H}$ represented by

$$\mathbb{R}^3 \ni (b, c, d) \quad \leftrightarrow \quad (b\mathbb{i} + c\mathbb{j} + d\mathbb{k}) \in su(2). \tag{2.18}$$

Thus, on the one hand, any Lie group element $N \in SU(2) \subset \mathbb{H}$ may be decomposed into

$$N = \mathsf{N} \cdot \mathsf{e}, \qquad \mathsf{N} \in S^3, \qquad \mathsf{e} = (\mathbb{1}, \mathbb{i}, \mathbb{j}, \mathbb{k}), \tag{2.19}$$

while, on the other hand, any Lie algebra element $r \in su(2)$ admits the decomposition

$$r = \boldsymbol{r} \cdot \boldsymbol{e}, \qquad \boldsymbol{r} \in \mathbb{R}^3, \qquad \boldsymbol{e} = (\mathbb{i}, \mathbb{j}, \mathbb{k}). \tag{2.20}$$

Multiplication of any two quaternions

$$A = \mathsf{A} \cdot \mathsf{e} = A_0 \mathbb{1} + \boldsymbol{A} \cdot \boldsymbol{e}, \qquad B = \mathsf{B} \cdot \mathsf{e} = B_0 \mathbb{1} + \boldsymbol{B} \cdot \boldsymbol{e} \tag{2.21}$$

then yields

$$AB^\dagger = (\mathsf{A} \cdot \mathsf{B})\mathbb{1} + (-\boldsymbol{A} \times \boldsymbol{B} + B_0 \boldsymbol{A} - A_0 \boldsymbol{B}) \cdot \boldsymbol{e} \tag{2.22}$$

so that, in particular,

$$XY = -(\boldsymbol{X} \cdot \boldsymbol{Y})\mathbb{1} + (\boldsymbol{X} \times \boldsymbol{Y}) \cdot \boldsymbol{e} \tag{2.23}$$

for any $su(2)$ matrices $X = \boldsymbol{X} \cdot \boldsymbol{e}$ and $Y = \boldsymbol{Y} \cdot \boldsymbol{e}$. The latter two relations are based on the fact that Hermitian conjugation of quaternions is represented by

$$A = A_0 \mathbb{1} + \boldsymbol{A} \cdot \boldsymbol{e}, \qquad A^\dagger = A_0 \mathbb{1} - \boldsymbol{A} \cdot \boldsymbol{e}. \tag{2.24}$$

The decompositions (2.20) and (2.23) now imply that the $su(2)$ version of the Lund–Regge equation (2.6) is given by

$$r_{xy} = \tfrac{1}{2}[r_x, r_y]. \tag{2.25}$$

The latter guarantees that there exists an $N \in SU(2)$ obeying the compatible linear system

$$N_x = r_x N, \qquad N_y = -r_y N. \tag{2.26}$$

Thus, by virtue of $N^\dagger N = \mathbb{1}$, we obtain the representation

$$r_x = N_x N^\dagger, \qquad r_y = -N_y N^\dagger \tag{2.27}$$

of the tangent vectors $\boldsymbol{r}_x$ and $\boldsymbol{r}_y$ to the Lund–Regge surface Σ. Cross-differentiation then leads to the matrix equation

$$\left(N N_x^\dagger\right)_y + \left(N N_y^\dagger\right)_x = 0 \tag{2.28}$$

or, equivalently,

$$(N^\dagger N_x)_y + (N^\dagger N_y)_x = 0. \tag{2.29}$$

The latter constitutes the standard form of the $SU(2)$ chiral model [16] and is equivalent to the $O(4)$ nonlinear σ-model (2.13) with the identification (2.19). Furthermore, the variant (2.28) demonstrates that $\tilde{N} = N^\dagger$ is another solution of the chiral model and, accordingly, a second Lund–Regge surface $\tilde{\Sigma}$ with position vector $\tilde{\boldsymbol{r}}$ is obtained by integration of the pair

$$\tilde{r}_x = N_x^\dagger N, \qquad \tilde{r}_y = -N_y^\dagger N. \tag{2.30}$$

In the following section, it is verified that the transition from Σ to $\tilde{\Sigma}$ via the chiral model corresponds to the invariance $\varphi \to -\varphi$ of the Pohlmeyer–Lund–Regge system (2.8). Moreover, it is shown that the Lund–Regge pairs $(\boldsymbol{r}, \tilde{\boldsymbol{r}})$ are naturally encoded in a generalization of the classical Lelieuvre formulae.

2.4. A generalization of the Lelieuvre formulae

In vector notation, the connection between the Lund–Regge equation and the $O(4)$ nonlinear σ-model encoded in (2.27) may be expressed as

$$\begin{aligned} \boldsymbol{r}_x &= \boldsymbol{N}\times\boldsymbol{N}_x + N_0\boldsymbol{N}_x - N_{0x}\boldsymbol{N}\\ \boldsymbol{r}_y &= \boldsymbol{N}_y\times\boldsymbol{N} + N_{0y}\boldsymbol{N} - N_0\boldsymbol{N}_y \end{aligned} \tag{2.31}$$

by virtue of the expansion (2.22). Similarly, the vector analogue of the pair (2.30) reads

$$\begin{aligned} \tilde{\boldsymbol{r}}_x &= \boldsymbol{N}\times\boldsymbol{N}_x - N_0\boldsymbol{N}_x + N_{0x}\boldsymbol{N}\\ \tilde{\boldsymbol{r}}_y &= \boldsymbol{N}_y\times\boldsymbol{N} - N_{0y}\boldsymbol{N} + N_0\boldsymbol{N}_y \end{aligned} \tag{2.32}$$

so that

$$\begin{aligned} \boldsymbol{r}_{xx} &= \boldsymbol{N}\times\boldsymbol{N}_{xx} + N_0\boldsymbol{N}_{xx} - N_{0xx}\boldsymbol{N}\\ \boldsymbol{r}_{xy} &= \boldsymbol{N}_y\times\boldsymbol{N}_x + N_{0y}\boldsymbol{N}_x - N_{0x}\boldsymbol{N}_y\\ \boldsymbol{r}_{yy} &= \boldsymbol{N}_{yy}\times\boldsymbol{N} + N_{0yy}\boldsymbol{N} - N_0\boldsymbol{N}_{yy} \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} \tilde{\boldsymbol{r}}_{xx} &= \boldsymbol{N}\times\boldsymbol{N}_{xx} - N_0\boldsymbol{N}_{xx} + N_{0xx}\boldsymbol{N}\\ \tilde{\boldsymbol{r}}_{xy} &= \boldsymbol{N}_y\times\boldsymbol{N}_x - N_{0y}\boldsymbol{N}_x + N_{0x}\boldsymbol{N}_y\\ \tilde{\boldsymbol{r}}_{yy} &= \boldsymbol{N}_{yy}\times\boldsymbol{N} - N_{0yy}\boldsymbol{N} + N_0\boldsymbol{N}_{yy} \end{aligned} \tag{2.34}$$

by virtue of the nonlinear σ-model (2.13). It is observed in passing that $\boldsymbol{N}$ constitutes a normal to the 'mid-surface' $\bar{\Sigma}$ defined by

$$\bar{\boldsymbol{r}} = \frac{\boldsymbol{r}+\tilde{\boldsymbol{r}}}{2} \tag{2.35}$$

since the relations (2.31), (2.32) enshrine the Lelieuvre formulae

$$\bar{\boldsymbol{r}}_x = \boldsymbol{N}\times\boldsymbol{N}_x, \qquad \bar{\boldsymbol{r}}_y = \boldsymbol{N}_y\times\boldsymbol{N}. \tag{2.36}$$

Hence, x and y represent asymptotic coordinates [15] on $\bar{\Sigma}$.

The above relations show that, in order to determine the fundamental forms associated with the surfaces Σ and $\tilde{\Sigma}$, it is now required to evaluate inner products of vectors of the form

$$\boldsymbol{V} = \boldsymbol{A}\times\boldsymbol{B} + \epsilon(A_0\boldsymbol{B} - B_0\boldsymbol{A}), \qquad \boldsymbol{W} = \boldsymbol{C}\times\boldsymbol{D} + \epsilon(C_0\boldsymbol{D} - D_0\boldsymbol{C}), \tag{2.37}$$

where $\epsilon = \pm 1$. Thus, a short calculation reveals that

$$\boldsymbol{V}\cdot\boldsymbol{W} = (\mathsf{A}\cdot\mathsf{C})(\mathsf{B}\cdot\mathsf{D}) - (\mathsf{A}\cdot\mathsf{D})(\mathsf{B}\cdot\mathsf{C}) + \epsilon|\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}|, \tag{2.38}$$

where $\mathsf{A} = (A_0, \boldsymbol{A}), \ldots, \mathsf{D} = (D_0, \boldsymbol{D})$, so that, on the one hand,

$$\boldsymbol{r}_x^2 = \tilde{\boldsymbol{r}}_x^2 = \mathsf{N}_x^2, \qquad \boldsymbol{r}_x\cdot\boldsymbol{r}_y = \tilde{\boldsymbol{r}}_x\cdot\tilde{\boldsymbol{r}}_y = -\mathsf{N}_x\cdot\mathsf{N}_y, \qquad \boldsymbol{r}_y^2 = \tilde{\boldsymbol{r}}_y^2 = \mathsf{N}_y^2. \tag{2.39}$$

Consequently, the surfaces Σ and $\tilde{\Sigma}$ are isometric. On the other hand, the relations

$$\boldsymbol{r}_{xx}\cdot\boldsymbol{r}_{xy} = -\tilde{\boldsymbol{r}}_{xx}\cdot\tilde{\boldsymbol{r}}_{xy}, \qquad \boldsymbol{r}_{xy}^2 = \tilde{\boldsymbol{r}}_{xy}^2, \qquad \boldsymbol{r}_{yy}\cdot\boldsymbol{r}_{xy} = -\tilde{\boldsymbol{r}}_{yy}\cdot\tilde{\boldsymbol{r}}_{xy} \tag{2.40}$$

demonstrate that the off-diagonal terms of the second fundamental forms of Σ and $\tilde{\Sigma}$ coincide while the diagonal terms exhibit opposite signs. It is evident that this corresponds to the invariance $\varphi \to -\varphi$ of the Pohlmeyer–Lund–Regge system (2.8) encoding the transition from Σ to $\tilde{\Sigma}$.

If the right-hand sides of the classical Lelieuvre formulae (2.11) are interpreted in the standard manner as $so(3)$ matrices then Lelieuvre-type formulae may be formulated for vectors of arbitrary dimension. In particular, if N is a solution of the $O(4)$ nonlinear σ-model (2.13) then the pair

$$R_x = 2\big(\mathsf{N}_x\mathsf{N}^{\mathrm{T}} - \mathsf{N}\mathsf{N}_x^{\mathrm{T}}\big), \qquad R_y = 2\big(\mathsf{N}\mathsf{N}_y^{\mathrm{T}} - \mathsf{N}_y\mathsf{N}^{\mathrm{T}}\big) \tag{2.41}$$

is compatible. Since the right-hand sides of these relations are anti-symmetric, we may assume that $R \in so(4)$. Accordingly, the generalized Lelieuvre formulae (2.41) are readily shown to imply that R is a solution of the $so(4)$ version of the Lund–Regge equation in the matrix form (2.25), that is

$$R_{xy} = \tfrac{1}{2}[R_x, R_y]. \tag{2.42}$$

If we now employ the Lie algebra isomorphism $so(4) \cong so(3) \oplus so(3)$ so that $so(4)$ may be decomposed into the direct sum of two $so(3)$ Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ with $[\mathfrak{g}, \tilde{\mathfrak{g}}] = \mathfrak{o}$ then the two components $r \in \mathfrak{g}$ and $\tilde{r} \in \tilde{\mathfrak{g}}$ of R obey the '$so(3)$-valued' Lund–Regge equations

$$r_{xy} = \tfrac{1}{2}[r_x, r_y], \qquad \tilde{r}_{xy} = \tfrac{1}{2}[\tilde{r}_x, \tilde{r}_y]. \tag{2.43}$$

Indeed, a canonical basis of $so(4)$ is given by the anti-symmetric matrices

$$\begin{gathered} L_1 = \begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}\\ \tilde{L}_1 = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tilde{L}_2 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \tilde{L}_3 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix} \end{gathered} \tag{2.44}$$

or, equivalently,

$$(L_i)_{\alpha\beta} = \epsilon_{0i\beta\alpha} + \delta_{i\alpha}\delta_{0\beta} - \delta_{i\beta}\delta_{0\alpha}, \qquad (\tilde{L}_i)_{\alpha\beta} = \epsilon_{0i\beta\alpha} - \delta_{i\alpha}\delta_{0\beta} + \delta_{i\beta}\delta_{0\alpha}, \tag{2.45}$$

where $\epsilon_{\alpha\beta\mu\nu}$ and $\delta_{\alpha\beta}$ denote the usual Levi-Civita and Kronecker symbols, respectively. It is readily verified that

$$[L_i, L_k] = 2\epsilon_{ikl}L_l, \qquad [L_i, \tilde{L}_k] = 0, \qquad [\tilde{L}_i, \tilde{L}_k] = 2\epsilon_{ikl}\tilde{L}_l \tag{2.46}$$

and hence the matrices L_i and $\tilde{L}_i$ are generators of the $so(3)$ subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. Decomposition of R into

$$R = \boldsymbol{r}\cdot\boldsymbol{L} + \tilde{\boldsymbol{r}}\cdot\tilde{\boldsymbol{L}} \tag{2.47}$$

with $\boldsymbol{L} = (L_1, L_2, L_3)$ and $\tilde{\boldsymbol{L}} = (\tilde{L}_1, \tilde{L}_2, \tilde{L}_3)$ then reveals that the generalized Lelieuvre formulae (2.41) are nothing but a compact reformulation of the relations (2.31), (2.32). Thus, the generalized Lelieuvre formulae relate the $O(4)$ nonlinear σ-model to pairs of Lund–Regge surfaces in the same manner as the classical Lelieuvre formulae encapsulate the connection between the $O(3)$ nonlinear σ-model and pseudospherical surfaces.

3. Discrete Chebyschev nets and integrable discrete Lund–Regge surfaces

In this section, it is demonstrated that the analysis undertaken in the preceding may be carried over to the discrete setting in such a manner that integrability is preserved. We begin with a discussion of discrete Chebyshev nets which have been used independently by Sauer [9] and Wunderlich [10] to define discrete analogues of pseudospherical surfaces. In fact, this connection with discrete surfaces of constant negative Gaußian curvature has provided the first indication of the importance of discrete Chebyschev nets in integrable discrete differential geometry (see [12] and references therein).

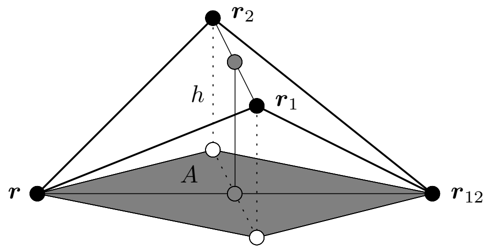

Figure 1. The geometry of a Chebyschev quadrilateral.

3.1. Discrete Chebyshev nets

In the following, we refer to quadrilateral lattices of the type

$$\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3, \qquad (n_1, n_2) \mapsto \boldsymbol{r}(n_1, n_2) \tag{3.1}$$

as discrete surfaces. A Chebyshev lattice or discrete Chebyshev net is defined by the property that opposite edges of any (non-planar) quadrilateral are of equal length. Thus, a discrete Chebyshev net is composed of 'skew' parallelograms. If the vertices of any quadrilateral are denoted by

$$\begin{aligned} &\boldsymbol{r} = \boldsymbol{r}(n_1, n_2), & &\boldsymbol{r}_1 = \boldsymbol{r}(n_1 + 1, n_2) \\ &\boldsymbol{r}_{12} = \boldsymbol{r}(n_1 + 1, n_2 + 1), & &\boldsymbol{r}_2 = \boldsymbol{r}(n_1, n_2 + 1) \end{aligned} \tag{3.2}$$

(cf figure 1) then discrete Chebyshev nets are encoded in the relations

$$\Delta_2(\Delta_1 \boldsymbol{r})^2 = 0, \qquad \Delta_1(\Delta_2 \boldsymbol{r})^2 = 0, \tag{3.3}$$

where the difference operators Δ_i are defined by $\Delta_i f = f_i - f$. Since the above relations may be formulated as

$$(\Delta_{12}\boldsymbol{r}) \cdot (\boldsymbol{r}_{12} - \boldsymbol{r}) = 0, \qquad (\Delta_{12}\boldsymbol{r}) \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1) = 0 \tag{3.4}$$

with $\Delta_{12} = \Delta_1\Delta_2 = \Delta_2\Delta_1$, discrete Chebyshev nets are governed by the difference equation

$$\Delta_{12}\boldsymbol{r} = \frac{\sigma}{2}(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1), \qquad \sigma = \sigma(n_1, n_2). \tag{3.5}$$

It is readily verified that, in the formal continuum limit $\boldsymbol{r}(x, y) = \boldsymbol{r}(\epsilon_1 n_1, \epsilon_2 n_2)$, $\epsilon_i \to 0$, the difference equation (3.5) reduces to the differential equation (2.4).

In geometric terms, if we define the 'discrete normal' to a skew parallelogram by

$$\hat{\boldsymbol{N}} = \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)|} \tag{3.6}$$

then relation (3.5) is equivalent to

$$\Delta_{12}\boldsymbol{r} \parallel \hat{\boldsymbol{N}} \tag{3.7}$$

and, as illustrated in figure 1, expresses the fact that the diagonals ($\boldsymbol{r}_{12} - \boldsymbol{r}$ and $\boldsymbol{r}_2 - \boldsymbol{r}_1$) of a skew parallelogram are orthogonal to the line segment $\left(\frac{1}{2}\Delta_{12}\boldsymbol{r}\right)$ connecting their centres. Moreover, the function σ may be regarded as a measure of the 'curvature' of the parallelograms since

$$\sigma = \frac{(\Delta_{12}\boldsymbol{r}) \cdot \hat{\boldsymbol{N}}}{\frac{1}{2}|(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)|} = \frac{2h}{A}, \tag{3.8}$$

where h and A denote the 'height' and 'projected area' of a skew parallelogram, respectively, as displayed in figure 1. It is observed that the representations (3.8) of the function σ are the canonical analogues of the expressions

$$\sigma = \frac{\boldsymbol{r}_{xy} \cdot \hat{\boldsymbol{N}}}{|\boldsymbol{r}_x \times \boldsymbol{r}_y|} = \frac{f}{\sqrt{\det \mathrm{I}}} \tag{3.9}$$

valid in the differential-geometric setting. Here, f is the off-diagonal coefficient of the second fundamental form.

3.2. Discrete Lund–Regge surfaces and nonlinear σ-models

It is evident that any function σ which tends to unity in the above-mentioned formal continuum limit is associated with a discretization of the Lund–Regge equation (2.6). For instance, the simplest choice $\sigma = 1$ corresponds to

$$\Delta_{12}\boldsymbol{r} = \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{2}. \tag{3.10}$$

It is natural to enquire as to whether there exists any connection with the standard integrable discretization of the $O(n)$ nonlinear σ-model [28]

$$\mathsf{N}_{12} + \mathsf{N} = \frac{\mathsf{N} \cdot (\mathsf{N}_1 + \mathsf{N}_2)}{1 + \mathsf{N}_1 \cdot \mathsf{N}_2}(\mathsf{N}_1 + \mathsf{N}_2), \qquad \mathsf{N}^2 = 1 \tag{3.11}$$

in the case $n = 4$. In this connection, it is enlightening to consider the particular case of the $O(3)$ nonlinear σ-model

$$\hat{\boldsymbol{N}}_{12} + \hat{\boldsymbol{N}} = \frac{\hat{\boldsymbol{N}} \cdot (\hat{\boldsymbol{N}}_1 + \hat{\boldsymbol{N}}_2)}{1 + \hat{\boldsymbol{N}}_1 \cdot \hat{\boldsymbol{N}}_2}(\hat{\boldsymbol{N}}_1 + \hat{\boldsymbol{N}}_2), \qquad \hat{\boldsymbol{N}}^2 = 1 \tag{3.12}$$

which is known to constitute the 'spherical representation' of discrete pseudospherical surfaces [8]. In fact, given a solution of the discrete nonlinear σ-model (3.12), the discrete Lelieuvre formulae

$$\Delta_1\boldsymbol{r} = \hat{\boldsymbol{N}} \times \hat{\boldsymbol{N}}_1, \qquad \Delta_2\boldsymbol{r} = \hat{\boldsymbol{N}}_2 \times \hat{\boldsymbol{N}} \tag{3.13}$$

are compatible and $\boldsymbol{r}$ constitutes the position vector of a discrete surface of constant negative Gaußian curvature. Multiplication of (3.12) by $\hat{\boldsymbol{N}}_1$ and $\hat{\boldsymbol{N}}_2$ respectively shows that

$$\Delta_2(\hat{\boldsymbol{N}}_1 \cdot \hat{\boldsymbol{N}}) = 0, \qquad \Delta_1(\hat{\boldsymbol{N}}_2 \cdot \hat{\boldsymbol{N}}) = 0 \tag{3.14}$$

and hence discrete pseudospherical surfaces constitute discrete Chebyshev nets since

$$(\Delta_1\boldsymbol{r})^2 = 1 - (\hat{\boldsymbol{N}}_1 \cdot \hat{\boldsymbol{N}})^2, \qquad (\Delta_2\boldsymbol{r})^2 = 1 - (\hat{\boldsymbol{N}}_2 \cdot \hat{\boldsymbol{N}})^2. \tag{3.15}$$

Specifically, it may be directly verified that the position vector $\boldsymbol{r}$ obeys the difference equation

$$\Delta_{12}\boldsymbol{r} = \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{\alpha(n_1) + \beta(n_2)} \tag{3.16}$$

with $\alpha = \hat{\boldsymbol{N}}_1 \cdot \hat{\boldsymbol{N}}$ and $\beta = \hat{\boldsymbol{N}}_2 \cdot \hat{\boldsymbol{N}}$ or, equivalently,

$$\alpha = \sqrt{1 - (\Delta_1\boldsymbol{r})^2}, \qquad \beta = \sqrt{1 - (\Delta_2\boldsymbol{r})^2}. \tag{3.17}$$

Here, in view of the formal continuum limit, we have made the admissible assumption that $\hat{\boldsymbol{N}}_i \cdot \hat{\boldsymbol{N}} > 0$ so that $\alpha + \beta \to 2$ as $\epsilon_i \to 0$. Accordingly, (3.16), (3.17) constitute a discretization of the Lund–Regge equation (2.6) which, as in the classical case, incorporates (discrete) pseudospherical surfaces.

3.3. Generalized discrete Lelieuvre formulae and a discrete chiral model

In the following section, it is established that the lattice equation (3.16) is integrable for *arbitrary* functions α and β depending on their respective arguments. Thus, in particular, the two discretizations (3.10) and (3.16), (3.17) admit Lax pairs and associated Bäcklund transformations. We now demonstrate that the latter discretization is but another avatar of the discrete $O(4)$ nonlinear σ-model. To this end, it is observed that the generalized discrete Lelieuvre formulae

$$\Delta_1 R = 2\big(\mathsf{N}_1\mathsf{N}^{\mathrm{T}} - \mathsf{N}\mathsf{N}_1^{\mathrm{T}}\big), \qquad \Delta_2 R = 2\big(\mathsf{N}\mathsf{N}_2^{\mathrm{T}} - \mathsf{N}_2\mathsf{N}^{\mathrm{T}}\big) \tag{3.18}$$

are compatible modulo the discrete $O(4)$ nonlinear σ-model (3.11) so that, once again, it may be assumed that $R \in so(4)$. A short calculation then reveals that

$$\Delta_{12} R = \frac{[R_{12} - R, R_2 - R_1]}{2\mathsf{N}\cdot(\mathsf{N}_1 + \mathsf{N}_2)} \tag{3.19}$$

so that decomposition of R into its $so(3) \oplus so(3)$ components $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ according to (2.47) leads to two difference equations of the form (3.16) since $\mathsf{N}_1 \cdot \mathsf{N}$ and $\mathsf{N}_2 \cdot \mathsf{N}$ are functions of n_1 and n_2, respectively. Moreover, the generalized discrete Lelieuvre formulae expressed in terms of $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$, that is

$$\begin{aligned}
\Delta_1 \boldsymbol{r} &= \boldsymbol{N} \times \boldsymbol{N}_1 + N_0 \boldsymbol{N}_1 - N_{01} \boldsymbol{N} \\
\Delta_2 \boldsymbol{r} &= \boldsymbol{N}_2 \times \boldsymbol{N} + N_{02} \boldsymbol{N} - N_0 \boldsymbol{N}_2 \\
\Delta_1 \tilde{\boldsymbol{r}} &= \boldsymbol{N} \times \boldsymbol{N}_1 - N_0 \boldsymbol{N}_1 + N_{01} \boldsymbol{N} \\
\Delta_2 \tilde{\boldsymbol{r}} &= \boldsymbol{N}_2 \times \boldsymbol{N} - N_{02} \boldsymbol{N} + N_0 \boldsymbol{N}_2,
\end{aligned} \tag{3.20}$$

imply that

$$(\Delta_1 \boldsymbol{r})^2 = (\Delta_1 \tilde{\boldsymbol{r}})^2 = 1 - (\mathsf{N}_1 \cdot \mathsf{N})^2, \qquad (\Delta_2 \boldsymbol{r})^2 = (\Delta_2 \tilde{\boldsymbol{r}})^2 = 1 - (\mathsf{N}_2 \cdot \mathsf{N})^2 \tag{3.21}$$

by virtue of the identity (2.38). We therefore conclude that any solution N of the discrete $O(4)$ nonlinear σ-model (3.11) corresponds to a pair of solutions $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ of the discrete Lund–Regge equation

$$\begin{aligned}
\Delta_{12} \boldsymbol{r} &= \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{\alpha(n_1) + \beta(n_2)} \\
\alpha &= \sqrt{1 - (\Delta_1 \boldsymbol{r})^2}, \qquad \beta = \sqrt{1 - (\Delta_2 \boldsymbol{r})^2}.
\end{aligned} \tag{3.22}$$

In order to prove equivalence, it remains to establish the 'converse' of the above statement. Thus, the $su(2)$ analogue of the discrete Lund–Regge equation (3.22) is given by

$$(\alpha + \beta)\Delta_{12} r = (r_{12} - r_2)(r_2 - r) - (r_{12} - r_1)(r_1 - r) \tag{3.23}$$

with $r = \boldsymbol{r} \cdot \boldsymbol{e}$. Indeed, the trace-free part of (3.23) is equivalent to the discrete Lund–Regge equation while its trace terms yield

$$(\Delta_{12} \boldsymbol{r}) \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1) = 0, \tag{3.24}$$

which is a consequence of $(3.22)_1$. On introduction of the matrices

$$U = \alpha \mathbb{1} + \Delta_1 r, \qquad V = \beta \mathbb{1} - \Delta_2 r, \tag{3.25}$$

the above matrix equation may be written as

$$U_2 V = V_1 U \tag{3.26}$$

which, in turn, guarantees the compatibility of the linear pair

$$N_1 = U N, \qquad N_2 = V N. \tag{3.27}$$

Here, we may assume that $N \in SU(2)$ since

$$U^\dagger U = \alpha^2 \mathbb{1} - (\Delta_1 r)^2 = [\alpha^2 + (\Delta_1 \boldsymbol{r})^2]\mathbb{1} = \mathbb{1} \tag{3.28}$$

and, similarly, $V^\dagger V = \mathbb{1}$ by virtue of the definitions (3.22)$_{2,3}$. Accordingly, the pair (3.27) may be formulated as

$$\Delta_1 r = N_1 N^\dagger - \alpha \mathbb{1}, \qquad \Delta_2 r = -N_2 N^\dagger + \beta \mathbb{1}, \tag{3.29}$$

which, on use of the decomposition $N = \mathsf{N} \cdot \mathsf{e} = N_0 \mathbb{1} + \boldsymbol{N} \cdot \boldsymbol{e}$ and the identity (2.22), is seen to be equivalent to the generalized discrete Lelieuvre formulae in the form (3.20) together with

$$\alpha = \mathsf{N}_1 \cdot \mathsf{N}, \qquad \beta = \mathsf{N}_2 \cdot \mathsf{N}. \tag{3.30}$$

Since the generalized discrete Lelieuvre formulae imply that N must be a solution of the discrete $O(4)$ nonlinear σ-model, the proof is complete.

In conclusion, it is noted that elimination of r from the pair (3.29) yields

$$N_{12}\left(N_1^\dagger + N_2^\dagger\right) = (N_1 + N_2)N^\dagger. \tag{3.31}$$

On the one hand, this matrix equation may be reformulated as

$$\left(N_1^\dagger + N_2^\dagger\right)N_{12} = N^\dagger(N_1 + N_2), \tag{3.32}$$

which, in the formal continuum limit, becomes the $SU(2)$ chiral model (2.29). Thus, (3.32) constitutes an integrable discretization of the $SU(2)$ chiral model or, equivalently, the quaternionic version of the discrete $O(4)$ nonlinear σ-model (3.11). On the other hand, the form

$$(N_1 + N_2)N_{12}^\dagger = N\left(N_1^\dagger + N_2^\dagger\right) \tag{3.33}$$

reveals that, as in the continuous case, $\tilde{N} = N^\dagger$ represents another solution of the discrete $SU(2)$ chiral model corresponding to another solution $\tilde{r}$ of the discrete $su(2)$ Lund–Regge equation (3.23) defined by the analogue of the pair (3.29). Moreover, the mid-surface $\bar{\Sigma}$ of the associated two discrete surfaces defined by

$$\bar{\boldsymbol{r}} = \frac{\boldsymbol{r} + \tilde{\boldsymbol{r}}}{2} \tag{3.34}$$

gives rise to the discrete Lelieuvre formulae

$$\Delta_1 \bar{\boldsymbol{r}} = \boldsymbol{N} \times \boldsymbol{N}_1, \qquad \Delta_2 \bar{\boldsymbol{r}} = \boldsymbol{N}_2 \times \boldsymbol{N} \tag{3.35}$$

by virtue of the generalized discrete Lelieuvre formulae (3.20). This implies that any vertex $\bar{\boldsymbol{r}}$ of $\bar{\Sigma}$ and its four neighbours are coplanar with $\boldsymbol{N}$ being an associated normal as illustrated in figure 2. Thus, by definition, discrete Lund–Regge mid-surfaces constitute discrete asymptotic nets. The latter have been used extensively in the construction of integrable discretizations of surfaces which may be naturally parametrized in terms of asymptotic coordinates (see [12] and references therein).

4. A Lax pair and a Bäcklund transformation for the generalized discrete Lund–Regge equation

In this section, it is demonstrated that the generalized discrete Lund–Regge equation

$$\Delta_{12}\boldsymbol{r} = \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{\alpha(n_1) + \beta(n_2)} \tag{4.1}$$

admits a Lax pair and an associated Bäcklund transformation. Moreover, it is shown that the constraints $\alpha = \beta = 1$ and (3.17) associated with the two discretizations (3.10) and (3.22)

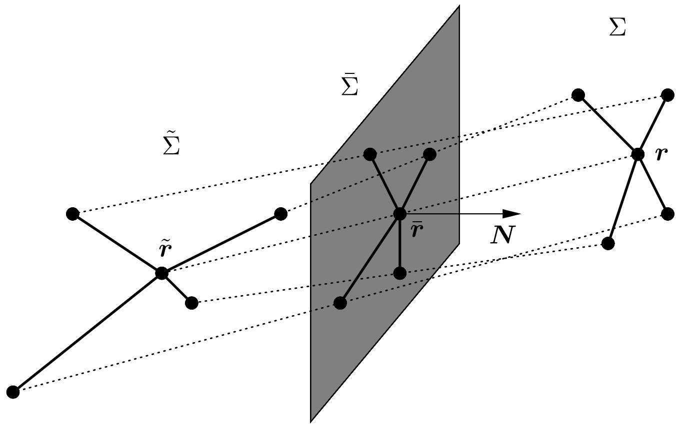

Figure 2. A planar star of a discrete Lund–Regge mid-surface.

respectively of the Lund–Regge equation are preserved by this Bäcklund transformation. It is noted that, in the formal continuum limit, the $su(2)$ version of the generalized discrete Lund–Regge equation becomes

$$r_{xy} = \frac{[r_x, r_y]}{\alpha(x) + \beta(y)}. \tag{4.2}$$

Hyperbolic and elliptic analogues of this matrix equation corresponding to various other Lie algebras/groups may be found in the literature, in particular, in the context of general relativity [17, 18].

4.1. A Lax pair

In order to reveal the fundamental nature of the generalized discrete Lund–Regge equation, we begin with the general form of a pair of linear equations

$$\Phi_1 = A(\lambda\mathbb{1} + U)\Phi, \qquad \Phi_2 = B(\lambda\mathbb{1} + V)\Phi \tag{4.3}$$

for a $\mathfrak{G}$-valued function Φ which is such that $\Phi_1\Phi^{-1}$ and $\Phi_2\Phi^{-1}$ are linear in a constant real parameter λ. It is recalled that $\mathfrak{G} = \{\Phi \in \mathbb{C}^{2,2} : \Phi^\dagger\Phi \sim \mathbb{1}\}$. The above pair is compatible if and only if the (λ-independent) matrix-valued functions A, B and U, V satisfy the polynomial equation

$$A_2(\lambda\mathbb{1} + U_2)B(\lambda\mathbb{1} + V) = B_1(\lambda\mathbb{1} + V_1)A(\lambda\mathbb{1} + U) \tag{4.4}$$

which is a consequence of the compatibility condition $\Phi_{12} = \Phi_{21}$. The terms quadratic in λ yield $A_2B = B_1A$ so that the matrices A and B may be absorbed into Φ by means of an appropriate gauge transformation $\Phi \to G\Phi$ which acts within $\mathfrak{G}$. Indeed, since, for instance, $A(\lambda\mathbb{1} + U) \in \mathfrak{G}$, the relation $(\lambda\mathbb{1} + U^\dagger)A^\dagger A(\lambda\mathbb{1} + U) \sim \mathbb{1}$ implies that $A \in \mathfrak{G}$. Thus, without loss of generality, we may assume that $A = B = \mathbb{1}$. The remaining powers of λ now give rise to

$$U_2 - U = V_1 - V, \qquad U_2V = V_1U. \tag{4.5}$$

In order to guarantee that $\Phi \in \mathfrak{G}$, it is required that

$$(\lambda\mathbb{1} + U^\dagger)(\lambda\mathbb{1} + U) \sim \mathbb{1}, \qquad (\lambda\mathbb{1} + V^\dagger)(\lambda\mathbb{1} + V) \sim \mathbb{1} \tag{4.6}$$

and hence $U, V \in \mathbb{H}$ by virtue of (2.16) so that

$$U = \alpha \mathbb{1} + u, \qquad V = -\beta \mathbb{1} + v, \qquad u, v \in su(2), \qquad \alpha, \beta \in \mathbb{R}. \tag{4.7}$$

Moreover, the compatibility condition (4.4) implies that

$$\det(\lambda \mathbb{1} + U_2) \det(\lambda \mathbb{1} + V) = \det(\lambda \mathbb{1} + V_1) \det(\lambda \mathbb{1} + U), \tag{4.8}$$

leading to

$$[(\lambda + \alpha_2)^2 + \det u_2][(\lambda - \beta)^2 + \det v] = [(\lambda - \beta_1)^2 + \det v_1][(\lambda + \alpha)^2 + \det u]. \tag{4.9}$$

The latter represents an algebraic system of equations for the scalars $\alpha, \beta, \alpha_2, \beta_1$ and $\det u, \det v, \det u_2, \det v_1$ which is readily seen to admit two solutions corresponding to

$$\alpha_2 = \alpha, \qquad \beta_1 = \beta \tag{4.10}$$

or $\alpha = -\beta, \alpha_2 = -\beta_1$. However, the analogue of the analysis presented below shows that the latter solution implies the former and hence we may assume that $\alpha = \alpha(n_1), \beta = \beta(n_2)$. Now, insertion of the parametrization (4.7) into the compatibility conditions (4.5) leads to

$$\alpha(v_1 - v) + \beta(u_2 - u) = u_2 v - v_1 u, \qquad u_2 - u = v_1 - v. \tag{4.11}$$

Thus, there exists an $su(2)$-valued potential r which parametrizes u and v according to

$$u = r_1 - r, \qquad v = r_2 - r \tag{4.12}$$

and the remaining equation $(4.11)_1$ becomes

$$(\alpha + \beta)\Delta_{12} r = (r_{12} - r_2)(r_2 - r) - (r_{12} - r_1)(r_1 - r) \tag{4.13}$$

which is indeed equivalent to the generalized discrete Lund–Regge equation (4.1) if the usual decomposition $r = \boldsymbol{r} \cdot \boldsymbol{e}$ is employed. The following theorem therefore obtains

Theorem 1. *The parameter-dependent linear pair*

$$\Phi_1 = (\lambda \mathbb{1} + U)\Phi, \qquad \Phi_2 = (\lambda \mathbb{1} + V)\Phi, \tag{4.14}$$

where $\Phi \in \mathfrak{G}$, is compatible if and only if the matrices U and V may be parametrized in terms of solutions of the generalized discrete Lund–Regge equation

$$\Delta_{12}\boldsymbol{r} = \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{\alpha(n_1) + \beta(n_2)} \tag{4.15}$$

according to

$$U = \alpha \mathbb{1} + (\boldsymbol{r}_1 - \boldsymbol{r}) \cdot \boldsymbol{e}, \qquad V = -\beta \mathbb{1} + (\boldsymbol{r}_2 - \boldsymbol{r}) \cdot \boldsymbol{e}. \tag{4.16}$$

In the terminology of soliton theory [3], the pair (4.14) constitutes a linear representation or a Lax pair for the generalized discrete Lund–Regge equation (4.15). Since $\Phi \in \mathfrak{G}$, it is gauge-equivalent to an '$SU(2)$-valued' Lax pair, the coefficients of which are, however, no longer linear in λ. It is shown below that the above Lax pair may be employed in the construction of a Bäcklund transformation for the generalized discrete Lund–Regge equation.

It is noted in passing that the standard integrability-preserving discretization of surfaces generated by a 'smoke ring' which evolves according to an integrable Heisenberg spin equation (cf section 5.4) [19] is encapsulated in the generalized discrete Lund–Regge equation via a particular choice of the functions $a(n_1), b(n_2)$ and the first integrals $(\Delta_1 \boldsymbol{r})^2 = f(n_1), (\Delta_2 \boldsymbol{r})^2 = g(n_2)$.

4.2. A Bäcklund transformation

We now seek a linear transformation of the 'eigenfunction' Φ which leaves form-invariant the linear representation (4.14) of the generalized discrete Lund–Regge equation. Thus, we make the ansatz

$$\Phi' = (\lambda \mathbb{1} + W)\Phi, \tag{4.17}$$

where W is a yet unspecified matrix-valued function. It is noted that the structure of the above transformation resembles that of the Lax pair. The implications of this observation will be discussed later. Insertion of Φ' into the primed version of (4.14), that is

$$\Phi'_1 = (\lambda \mathbb{1} + U')\Phi', \qquad \Phi'_2 = (\lambda \mathbb{1} + V')\Phi', \tag{4.18}$$

produces the system

$$\begin{aligned} W_1 - W &= U' - U, \qquad & W_1 U &= U'W \\ W_2 - W &= V' - V, \qquad & W_2 V &= V'W \end{aligned} \tag{4.19}$$

which, in turn, guarantees the compatibility of the pair (4.18). Accordingly, the matrices U' and V' obey the primed version of the system (4.5) and hence provide another solution of the generalized discrete Lund–Regge equation as long as $\Phi' \in \mathfrak{G}$. Alternatively, one may directly verify that U' and V' as defined by $(4.19)_{1,3}$ indeed satisfy the primed version of (4.5). Now, elimination of U' and V' from (4.19) leads to the two nonlinear equations

$$W_1(U - W) = (U - W)W, \qquad W_2(V - W) = (V - W)W \tag{4.20}$$

for the matrix W. Under the assumption that $U - W$ and $V - W$ are invertible, it may be shown that the compatibility condition $W_{12} = W_{21}$ is satisfied and hence W is uniquely determined by its value W_0 at some lattice point, say, $\boldsymbol{n}_0 = (0, 0)$. Moreover, if $W_0 \in \mathbb{H}$ then $W \in \mathbb{H}$ and $\Phi' \in \mathfrak{G}$ as required.

It turns out that the system (4.20) may, in fact, be linearized. In order to make good this assertion, we first observe that

$$W = -\Psi\Lambda\Psi^{-1}, \qquad \Lambda = \operatorname{diag}(\lambda_{[1]}, \lambda_{[2]}), \tag{4.21}$$

where $\lambda_{[i]}$ are constant parameters and Ψ obeys the linear pair

$$\Psi_1 = \Psi\Lambda + U\Psi, \qquad \Psi_2 = \Psi\Lambda + V\Psi, \tag{4.22}$$

constitutes a solution of the compatible system (4.20). Moreover, Ψ is uniquely determined by its value Ψ_0 at $\boldsymbol{n}_0$. If we choose W_0 in such a way that there exist two linearly independent eigenvectors of W_0 then $-\lambda_{[i]}$ represent the eigenvalues of W_0 and the columns of Ψ_0 are the corresponding eigenvectors. Hence, W indeed admits the parametrization (4.21). In conclusion, it is noted that the columns of Ψ are merely vector-valued solutions of the Lax pair (4.14) associated with the (complex) parameters $\lambda_{[i]}$. This confirms the compatibility of both (4.22) and (4.20).

If $W_0 \sim \mathbb{1}$ then $W = W_0$ and (4.17) essentially reduces to the identity transformation. If $W_0 \in \mathbb{H}$ and $W_0 \not\sim \mathbb{1}$ then W_0 has two distinct complex conjugate eigenvalues. Thus, in the context of the generalized discrete Lund–Regge equation corresponding to $W \in \mathbb{H}$, (4.21) represents the general solution of the system (4.20). Furthermore, if ψ is a vector-valued solution of the Lax pair (4.14) associated with the parameter $\lambda_{[1]} = \mu$ then, by virtue of the characterization $(2.16)_1$, $M\bar{\psi}$ constitutes a second solution with $\lambda_{[2]} = \bar{\mu}$. We may therefore make the choice

$$\Psi = (\psi, M\bar{\psi}) \in \mathbb{H}, \qquad \Lambda = \operatorname{diag}(\mu, \bar{\mu}) \in \mathbb{H} \tag{4.23}$$

in order to formulate a Bäcklund transformation [7] for the generalized discrete Lund–Regge equation.

Theorem 2. *The linear representation (4.14) of the generalized discrete Lund–Regge equation is form-invariant under the transformation*

$$\begin{aligned}
&\Phi \to \Phi' = (\lambda \mathbb{1} - \Psi\Lambda\Psi^{-1})\Phi\\
&r \to r' = r - \Psi\Lambda\Psi^{-1} + \Lambda\\
&\alpha \to \alpha' = \alpha\\
&\beta \to \beta' = \beta
\end{aligned} \tag{4.24}$$

with $r = \boldsymbol{r} \cdot \boldsymbol{e}$ and

$$\Psi = (\psi, M\bar{\psi}), \qquad \Lambda = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.25}$$

where ψ is a vector-valued solution of (4.14) corresponding to an arbitrary complex parameter μ. The solution $\boldsymbol{r}'$ of the generalized discrete Lund–Regge equation (4.15) obtained via the decomposition $r' = \boldsymbol{r}' \cdot \boldsymbol{e}$ admits the first integrals

$$(\Delta_1 \boldsymbol{r}')^2 = (\Delta_1 \boldsymbol{r})^2, \qquad (\Delta_2 \boldsymbol{r}')^2 = (\Delta_2 \boldsymbol{r})^2. \tag{4.26}$$

The constraints $\alpha = \beta = 1$ and (3.17) associated with the two discretizations (3.10) and (3.16), (3.17) respectively of the Lund–Regge equation are preserved.

Proof. The difference equations (4.20) imply that both $\mathrm{tr}W$ and $\det W$ are constant. Accordingly, system (4.19) gives rise to

$$\mathrm{tr}U' = \mathrm{tr}U, \qquad \mathrm{tr}V' = \mathrm{tr}V, \qquad \det U' = \det U, \qquad \det V' = \det V. \tag{4.27}$$

Since, by construction, U' and V' may be decomposed into

$$U' = \alpha' \mathbb{1} + \Delta_1 r, \qquad V' = \beta' \mathbb{1} + \Delta_2 r, \qquad r' = \boldsymbol{r}' \cdot \boldsymbol{e} \in su(2), \tag{4.28}$$

we conclude that $\alpha' = \alpha, \beta' = \beta$ and that the first integrals (4.26) hold. Any prescribed relations of the type $\alpha = \alpha[(\Delta_1 \boldsymbol{r})^2]$ and $\beta = \beta[(\Delta_2 \boldsymbol{r})^2]$ are therefore preserved. Moreover, the transformation laws $(4.19)_{1,3}$ may be brought into the form

$$\Delta_1(r' - r - W) = 0, \qquad \Delta_2(r' - r - W) = 0, \tag{4.29}$$

which shows that r' is indeed given by $(4.24)_2$, wherein the trace-free 'constant of summation' has been neglected without loss of generality. □

4.3. Three-dimensional (generalized) Lund–Regge lattices

Iterative application of the above Bäcklund transformation leads to a sequence of generalized discrete Lund–Regge surfaces with associated position vectors

$$\boldsymbol{r} = \boldsymbol{r}(n_1, n_2; n_3), \tag{4.30}$$

where n_3 labels the number of iterations. At each step, one may arbitrarily choose the Bäcklund parameter μ so that the collection of Bäcklund parameters may be encapsulated in the notation

$$\mu = \mu(n_3). \tag{4.31}$$

Thus, the Bäcklund transformation generates three-dimensional lattices

$$\boldsymbol{r} : \mathbb{Z}^3 \to \mathbb{R}^3 \tag{4.32}$$

which are such that any two-dimensional sublattice $\boldsymbol{r}(n_3 = \mathrm{const})$ constitutes a generalized discrete Lund–Regge surface. Since the Bäcklund transformation preserves the constraints which are associated with discrete Lund–Regge surfaces, lattices which consist of an infinite number of discrete Lund–Regge surfaces may also be constructed. In the following, we study in more detail the properties of these (generalized) 'Lund–Regge lattices'.

We begin with the requirement that a $\mathfrak{G}$-valued function $\Phi(n_1, n_2, n_3)$ satisfies a linear triad of the form

$$\Phi_1 = (\lambda \mathbb{1} + U)\Phi, \qquad \Phi_2 = (\lambda \mathbb{1} + V)\Phi, \qquad \Phi_3 = (\lambda \mathbb{1} + W)\Phi, \tag{4.33}$$

where λ is a constant real parameter and the subscript $_3$ designates a unit increment of the discrete variable n_3. The compatibility conditions $\Phi_{ik} = \Phi_{ki}$ then produce the system

$$\begin{aligned} U_2 - U &= V_1 - V, & U_2 V &= V_1 U \\ V_3 - V &= W_2 - W, & V_3 W &= W_2 V \\ W_1 - W &= U_3 - U, & W_1 U &= U_3 W \end{aligned} \tag{4.34}$$

which may be solved for U_2, U_3, V_3, V_1 and W_1, W_2. From an algebraic point of view, the above system is identical to the systems (4.5) and (4.19) with the prime being identified with the subscript $_3$. Consequently, it has already been demonstrated that the compatibility condition $W_{12} = W_{21}$ holds. For reasons of symmetry, the remaining compatibility conditions $U_{23} = U_{32}$ and $V_{13} = V_{31}$ are also satisfied. Thus, it emerges that the pairs U, V, V, W and W, U encapsulate three one-parameter families of generalized discrete Lund–Regge surfaces. Specifically, if we introduce the parametrization

$$U = a\mathbb{1} + u, \qquad V = b\mathbb{1} + v, \qquad W = c\mathbb{1} + w \tag{4.35}$$

with $a, b, c \in \mathbb{R}$ and $u, v, w \in su(2)$ then $(4.34)_{1,3,5}$ implies that there exists an $r \in su(2)$ such that

$$u = r_1 - r, \qquad v = r_2 - r, \qquad w = r_3 - r. \tag{4.36}$$

In addition, it is evident that $a = a(n_1)$, $b = b(n_2)$ and $c = c(n_3)$. Evaluation of the remaining conditions $(4.34)_{2,4,6}$ with $r = \boldsymbol{r} \cdot \boldsymbol{e}$ then gives rise to the following theorem.

Theorem 3. *The three copies*

$$\begin{aligned} \Delta_{12}\boldsymbol{r} &= \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{a(n_1) - b(n_2)} \\ \Delta_{23}\boldsymbol{r} &= \frac{(\boldsymbol{r}_{23} - \boldsymbol{r}) \times (\boldsymbol{r}_3 - \boldsymbol{r}_2)}{b(n_2) - c(n_3)} \\ \Delta_{31}\boldsymbol{r} &= \frac{(\boldsymbol{r}_{31} - \boldsymbol{r}) \times (\boldsymbol{r}_1 - \boldsymbol{r}_3)}{c(n_3) - a(n_1)} \end{aligned} \tag{4.37}$$

of the generalized discrete Lund–Regge equation are compatible. The Cauchy data

$$\boldsymbol{r}(n_1, 0, 0), \qquad \boldsymbol{r}(0, n_2, 0), \qquad \boldsymbol{r}(0, 0, n_3) \tag{4.38}$$

uniquely determine a 'Chebyschev lattice' $\boldsymbol{r}$, the two-dimensional sublattices of which constitute generalized discrete Lund–Regge surfaces.

It is remarked that for any given local Cauchy data $\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$ the triad (4.37) determines the lattice points $\boldsymbol{r}_{12}, \boldsymbol{r}_{23}, \boldsymbol{r}_{31}$ and, even though there exist three different ways of constructing the lattice point $\boldsymbol{r}_{123}$ corresponding to the application of any of the above-generalized discrete Lund–Regge equations, $\boldsymbol{r}_{123}$ is well defined. Thus, the above theorem essentially states that it is consistent to demand that the generalized discrete Lund–Regge equation holds on any face of the $\mathbb{Z}^3$ lattice. This procedure of imposing a two-dimensional lattice equation on the sublattices of a three-dimensional lattice has come to be known as the 'consistency approach' and may be used in the detection of integrability in certain classes of difference equations (cf [20]–[23]). It turns out that, in the current situation, the formal continuum limit respects consistency.

Corollary 1. *The three copies*

$$\begin{aligned} \boldsymbol{r}_{xy} &= 2\frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{a(x) - b(y)} \\ \boldsymbol{r}_{yz} &= 2\frac{\boldsymbol{r}_y \times \boldsymbol{r}_z}{b(y) - c(z)} \\ \boldsymbol{r}_{zx} &= 2\frac{\boldsymbol{r}_z \times \boldsymbol{r}_x}{c(z) - a(x)} \end{aligned} \tag{4.39}$$

of the generalized Lund–Regge equation are compatible. In particular, it is consistent to demand that the coordinate surfaces of a coordinate system in $\mathbb{R}^3$ *constitute Lund–Regge surfaces (corresponding to constants* a, b, c*).*

The above corollary is evidently valid since differentiation of $(4.39)_1$ with respect to z yields

$$\boldsymbol{r}_{xyz} = 4\frac{(\boldsymbol{r}_y \cdot \boldsymbol{r}_z)\boldsymbol{r}_x}{(a-b)(c-a)} + 4\frac{(\boldsymbol{r}_z \cdot \boldsymbol{r}_x)\boldsymbol{r}_y}{(b-c)(a-b)} + 4\frac{(\boldsymbol{r}_x \cdot \boldsymbol{r}_y)\boldsymbol{r}_z}{(c-a)(b-c)} \tag{4.40}$$

which is symmetric in x, y, z. Moreover, on application of the scaling

$$\begin{aligned} &\Phi \to \tilde{a}(n_1)\tilde{b}(n_2)\tilde{c}(n_3)\Phi \\ &\tilde{a}_1 = (\lambda + a)\tilde{a}, \qquad \tilde{b}_2 = (\lambda + b)\tilde{b}, \qquad \tilde{c}_3 = (\lambda + c)\tilde{c}, \end{aligned} \tag{4.41}$$

the linear triad (4.33) becomes

$$\Delta_1\Phi = \frac{\Delta_1 r}{\lambda + a}\Phi, \qquad \Delta_2\Phi = \frac{\Delta_2 r}{\lambda + b}\Phi, \qquad \Delta_3\Phi = \frac{\Delta_3 r}{\lambda + c}\Phi \tag{4.42}$$

so that

$$\Phi_x = \frac{r_x}{\lambda + a}\Phi, \qquad \Phi_y = \frac{r_y}{\lambda + b}\Phi, \qquad \Phi_z = \frac{r_z}{\lambda + c}\Phi \tag{4.43}$$

in the formal continuum limit. The latter is compatible modulo the $su(2)$ version of the system (4.31). It may now be directly verified that Φ obeys the λ-independent triad

$$\begin{aligned} \Phi_{xy} &= \frac{r_y}{b-a}\Phi_x + \frac{r_x}{a-b}\Phi_y \\ \Phi_{yz} &= \frac{r_z}{c-b}\Phi_y + \frac{r_y}{b-c}\Phi_z \\ \Phi_{zx} &= \frac{r_x}{a-c}\Phi_z + \frac{r_z}{c-a}\Phi_x \end{aligned} \tag{4.44}$$

and that this system is compatible without reference to the first-order triad (4.43). Thus, the generalized Lund–Regge system (4.39) represents a particular reduction of the Zakharov–Manakov system [24] which is obtained by evaluating the compatibility conditions $\Phi_{xyz} = \Phi_{yzx} = \Phi_{zxy}$ associated with the linear triad (4.44) for *a priori* arbitrary matrix-valued coefficients. The Zakharov–Manakov system may be regarded as a three-dimensional integrable generalization of the 'principal chiral field equation' and constitutes a matrix extension of the classical Darboux system governing conjugate coordinate systems in $\mathbb{R}^3$ [25].

It turns out that, at the discrete level, Φ obeys the λ-independent linear triad

$$\begin{aligned}\Delta_{12}\Phi &= \frac{\Delta_2 r_1}{b-a}\Delta_1\Phi + \frac{\Delta_1 r_2}{a-b}\Delta_2\Phi \\ \Delta_{23}\Phi &= \frac{\Delta_3 r_2}{c-b}\Delta_2\Phi + \frac{\Delta_2 r_3}{b-c}\Delta_3\Phi \\ \Delta_{31}\Phi &= \frac{\Delta_1 r_3}{a-c}\Delta_3\Phi + \frac{\Delta_3 r_1}{c-a}\Delta_1\Phi\end{aligned} \tag{4.45}$$

which is, once again, compatible modulo the generalized discrete Lund–Regge system (4.37). Accordingly, the latter constitutes a particular reduction of the discrete Zakharov–Manakov system set down in [26].

5. A universal permutability theorem

We now demonstrate that generalized discrete Lund–Regge surfaces may also be generated by means of standard matrix Darboux transformations applied to linear ordinary differential equations based on the $su(2)$ Lie algebra. In fact, we show that the generalized discrete Lund–Regge equation regarded as a four-point relation is *universal* in the sense that it constitutes the permutability theorem for any (1+1)-dimensional soliton system which admits an $su(2)$ linear representation amenable to the Darboux matrix method. Such systems include the nonlinear Schrödinger, Heisenberg spin, sine-Gordon, modified Korteweg-de Vries and Pohlmeyer–Lund–Regge equations. The latter encapsulate the self-induced transparency (SIT) and stimulated Raman scattering (SRS) equations and the Maxwell Bloch system (see, e.g., [7] and references therein).

5.1. The Darboux matrix method

It is well known [7] that the polynomial structure of any linear ordinary differential equation of the form

$$\Phi_x = g(\lambda)\Phi, \tag{5.1}$$

where g is an $su(2)$-valued polynomial in a real parameter λ and $\Phi \in \mathfrak{G}$, is preserved by the matrix Darboux transformation

$$\mathcal{D}_{[1]} : \Phi_{(1)} = P_{[1]}\left(\lambda\mathbb{1} - \Psi_{[1]}\Lambda_{[1]}\Psi_{[1]}^{-1}\right)\Phi. \tag{5.2}$$

Here, $P_{[1]}$ constitutes an arbitrary $SU(2)$-valued function and

$$\Psi_{[1]} = (\psi_{[1]}, M\bar{\psi}_{[1]}), \qquad \Lambda_{[1]} = \begin{pmatrix}\lambda_{[1]} & 0\\ 0 & \bar{\lambda}_{[1]}\end{pmatrix}, \qquad M = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{5.3}$$

where $\psi_{[1]}$ represents a vector-valued solution of the linear equation (5.1) corresponding to an arbitrary complex parameter $\lambda_{[1]}$. It is noted that the factor $(\lambda\mathbb{1} - \Psi_{[1]}\Lambda_{[1]}\Psi_{[1]}^{-1})$ in the above Darboux matrix linking Φ and $\Phi_{[1]}$ is identical in structure to that employed in the discrete matrix Darboux transformation (4.24).

Another Darboux transform $\Phi_{(2)} = \mathcal{D}_{[2]}(\Phi)$ may be generated from Φ by means of a vector-valued eigenfunction $\psi_{[2]}$ corresponding to a parameter $\lambda_{[2]}$ according to

$$\mathcal{D}_{[2]} : \quad \Phi_{(2)} = P_{[2]}\left(\lambda\mathbb{1} - \Psi_{[2]}\Lambda_{[2]}\Psi_{[2]}^{-1}\right)\Phi. \tag{5.4}$$

Particular vector-valued solutions of the transformed linear equations

$$\Phi_{(1)x} = g_{(1)}(\lambda)\Phi_{(1)}, \qquad \Phi_{(2)x} = g_{(2)}(\lambda)\Phi_{(2)} \tag{5.5}$$

$$
\begin{tikzcd}
& \Phi_{(2)} \arrow[dr, "\lambda_{[1]}"] & \\
\Phi \arrow[ur, "\lambda_{[2]}"] \arrow[dr, "\lambda_{[1]}"'] & & \Phi_{(12)} = \Phi_{(21)} \\
& \Phi_{(1)} \arrow[ur, "\lambda_{[2]}"'] &
\end{tikzcd}
$$

Figure 3. A commutative Bianchi diagram.

are then given by $\psi_{[2](1)} = \mathcal{D}_{[1]}(\psi_{[2]})$ and $\psi_{[1](2)} = \mathcal{D}_{[2]}(\psi_{[1]})$ with associated quantities

$$
\begin{aligned}
\Psi_{[2](1)} &= P_{[1]}\left(\Lambda_{[2]} - \Psi_{[1]}\Lambda_{[1]}\Psi_{[1]}^{-1}\right)\Psi_{[2]} \\
\Psi_{[1](2)} &= P_{[2]}\left(\Lambda_{[1]} - \Psi_{[2]}\Lambda_{[2]}\Psi_{[2]}^{-1}\right)\Psi_{[1]}.
\end{aligned}
\tag{5.6}
$$

The latter give rise to the compound Darboux transforms $\Phi_{(12)} = \mathcal{D}^*_{[2]}(\mathcal{D}_{[1]}(\Phi))$ and $\Phi_{(21)} = \mathcal{D}^*_{[1]}(\mathcal{D}_{[2]}(\Phi))$ defined by

$$
\begin{aligned}
\Phi_{(12)} &= P^*_{[2]}\big(\lambda\mathbb{1} - \Psi_{[2](1)}\Lambda_{[2]}\Psi_{[2](1)}^{-1}\big)\Phi_{(1)} \\
\Phi_{(21)} &= P^*_{[1]}\big(\lambda\mathbb{1} - \Psi_{[1](2)}\Lambda_{[1]}\Psi_{[1](2)}^{-1}\big)\Phi_{(2)},
\end{aligned}
\tag{5.7}
$$

where the matrices $P^*_{[1]}$ and $P^*_{[2]}$ in the matrix Darboux transformations $\mathcal{D}^*_{[1]}$ and $\mathcal{D}^*_{[2]}$ do not necessarily have to coincide with $P_{[1]}$ and $P_{[2]}$, respectively. The 'permutability theorem' [7]

$$
\mathcal{D}^*_{[2]} \circ \mathcal{D}_{[1]} = \mathcal{D}^*_{[1]} \circ \mathcal{D}_{[2]}
\tag{5.8}
$$

now states that $\Phi_{(12)} = \Phi_{(21)}$ provided that

$$
P^*_{[2]}P_{[1]} = P^*_{[1]}P_{[2]}.
\tag{5.9}
$$

This is encapsulated in the 'commutative Bianchi diagram' displayed in figure 3.

As an illustration, we consider the AKNS scattering problem

$$
\Phi_x = (g^0 + \lambda g^1)\Phi = \frac{1}{2}\begin{pmatrix} \mathrm{i}\lambda & q \\ -\bar{q} & -\mathrm{i}\lambda \end{pmatrix}\Phi
\tag{5.10}
$$

associated with the nonlinear Schrödinger hierarchy [3]. As indicated above, the matrix Darboux transformation $\mathcal{D}_{[1]}$ leaves invariant its polynomial structure for any matrix $P_{[1]}$. Moreover, it is well known (see, e.g., [7]) that the choice $P_{[1]} = \mathbb{1}$ guarantees that the particular form of the matrices g^0 and g^1 is preserved. Since any 'time evolution' of the eigenfunction Φ leading via compatibility to a particular member of the nonlinear Schrödinger hierarchy is likewise preserved, the matrix Darboux transformation $\mathcal{D}_{[1]}$ induces a Bäcklund transformation for all members of that hierarchy. For instance, the time evolution

$$
\Phi_t = \frac{1}{2}\begin{pmatrix} \mathrm{i}\big(\frac{1}{2}|q|^2 - \lambda^2\big) & \mathrm{i}q_x - \lambda q \\ \mathrm{i}\bar{q}_x + \lambda\bar{q} & -\mathrm{i}\big(\frac{1}{2}|q|^2 - \lambda^2\big) \end{pmatrix}\Phi
\tag{5.11}
$$

is compatible with the scattering problem (5.9) if and only if q is a solution of the nonlinear Schrödinger equation

$$
\mathrm{i}q_t + q_{xx} + \tfrac{1}{2}|q|^2 q = 0
\tag{5.12}
$$

and hence $\mathcal{D}_{[1]}$ generates a Bäcklund transform $q_{(1)}$ from q. Furthermore, the admissible choice $P_{[2]} = P^*_{[1]} = P^*_{[2]} = \mathbb{1}$ ensures that the permutability theorem (5.8) is valid. This implies, in turn, that the permutability theorem is applicable at the level of the solutions of the nonlinear Schrödinger hierarchy.

5.2. A universal permutability theorem

In order to analyse further the permutability theorem (5.8), it is convenient to introduce the matrices

$$U = -\Psi_{[1]}\Lambda_{[1]}\Psi_{[1]}^{-1}, \qquad U_{(2)} = -\tilde{\Psi}_{[1](2)}\Lambda_{[1]}\tilde{\Psi}_{[1](2)}^{-1}$$
$$V = -\Psi_{[2]}\Lambda_{[2]}\Psi_{[2]}^{-1}, \qquad V_{(1)} = -\tilde{\Psi}_{[2](1)}\Lambda_{[2]}\tilde{\Psi}_{[2](1)}^{-1} \tag{5.13}$$

with the definitions

$$\tilde{\Psi}_{[2](1)} = \left(\Lambda_{[2]} - \Psi_{[1]}\Lambda_{[1]}\Psi_{[1]}^{-1}\right)\Psi_{[2]}$$
$$\tilde{\Psi}_{[1](2)} = \left(\Lambda_{[1]} - \Psi_{[2]}\Lambda_{[2]}\Psi_{[2]}^{-1}\right)\Psi_{[1]}. \tag{5.14}$$

On use of (5.9), the commutativity property $\Phi_{(12)} = \Phi_{(21)}$ then reads

$$(\lambda\mathbb{1} + U_{(2)})(\lambda\mathbb{1} + V) = (\lambda\mathbb{1} + V_{(1)})(\lambda\mathbb{1} + U) \tag{5.15}$$

which is independent of the matrices $P_{[i]}$ and $P_{[i]}^*$. Now, on the one hand, explicit evaluation by means of (5.13) and (5.14) shows that (5.15) is indeed identically satisfied. On the other hand, since U, V and $U_{(2)}$, $V_{(1)}$ are independent of λ, (5.15) decomposes into the analogue of the system (4.5), that is

$$U_{(2)} - U = V_{(1)} - V, \qquad U_{(2)}V = V_{(1)}U. \tag{5.16}$$

If we parametrize the quaternions U, V and $U_{(2)}$, $V_{(1)}$ according to

$$U = a\mathbb{1} + u, \qquad V = b\mathbb{1} + v, \qquad U_{(2)} = a\mathbb{1} + u_{(2)}, \qquad V_{(1)} = b\mathbb{1} + v_{(1)}, \tag{5.17}$$

where $u, v, u_{(2)}, v_{(1)} \in su(2)$ and

$$a = -\mathrm{Re}(\lambda_{[1]}), \qquad b = -\mathrm{Re}(\lambda_{[2]}), \tag{5.18}$$

then (5.16) becomes

$$a(v_{(1)} - v) - b(u_{(2)} - u) = u_{(2)}v - v_{(1)}u, \qquad u_{(2)} - u = v_{(1)} - v. \tag{5.19}$$

The latter of these two equations implies that u, v and $u_{(2)}$, $v_{(1)}$ may be written as

$$u = r_{(1)} - r, \qquad v = r_{(2)} - r, \qquad u_{(2)} = r_{(12)} - r_{(2)}, \qquad v_{(1)} = r_{(12)} - r_{(1)}, \tag{5.20}$$

where $r, r_{(1)}, r_{(2)}$ and $r_{(12)}$ are $su(2)$-valued functions. The remaining equation expressed in terms of the vectors $\boldsymbol{r}, \boldsymbol{r}_{(1)}, \boldsymbol{r}_{(2)}, \boldsymbol{r}_{(12)}$ defined by

$$r = \boldsymbol{r}\cdot\boldsymbol{e}, \qquad r_{(1)} = \boldsymbol{r}_{(1)}\cdot\boldsymbol{e}, \qquad r_{(2)} = \boldsymbol{r}_{(2)}\cdot\boldsymbol{e}, \qquad r_{(12)} = \boldsymbol{r}_{(12)}\cdot\boldsymbol{e} \tag{5.21}$$

then gives rise to the following key theorem:

Theorem 4. *The permutability theorem (5.8) associated with the matrix Darboux transformation for any linear equation of the kind (5.1) is encapsulated in the four-point relation of 'Lund–Regge type'*

$$\boldsymbol{r}_{(12)} - \boldsymbol{r}_{(1)} - \boldsymbol{r}_{(2)} + \boldsymbol{r} = \frac{(\boldsymbol{r}_{(12)} - \boldsymbol{r}) \times (\boldsymbol{r}_{(2)} - \boldsymbol{r}_{(1)})}{a - b}, \tag{5.22}$$

where

$$a = -\mathrm{Re}(\lambda_{[1]}), \qquad b = -\mathrm{Re}(\lambda_{[2]}) \tag{5.23}$$

and

$$(\boldsymbol{r}_{(1)} - \boldsymbol{r})^2 = [\mathrm{Im}(\lambda_{[1]})]^2, \qquad (\boldsymbol{r}_{(2)} - \boldsymbol{r})^2 = [\mathrm{Im}(\lambda_{[2]})]^2. \tag{5.24}$$

In order to illustrate the applicability of the above theorem, we return to the scattering problem (5.10) of the nonlinear Schrödinger hierarchy. For brevity, we focus on the compatible time evolution (5.11) associated with the nonlinear Schrödinger equation. Thus, insertion of the Darboux transform $\Phi_{(1)}$ as given by (5.2) into the transformed scattering problem (5.5) leads to the relation

$$Q_{(1)} = Q - [u, \Bbbk], \qquad Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \qquad u = \mathrm{Im}(\lambda_{[1]})\Psi_{[1]}\Bbbk\Psi_{[1]}^{-1} \tag{5.25}$$

between the solutions q and $q_{(1)}$ of the nonlinear Schrödinger equation (5.12). In terms of the ratio

$$\xi_{[1]} = \frac{\psi_{[1]}^1}{\psi_{[1]}^2}, \qquad \psi_{[1]} = \begin{pmatrix} \psi_{[1]}^1 \\ \psi_{[1]}^2 \end{pmatrix}, \tag{5.26}$$

this relation becomes

$$q_{(1)} = q - 4\,\mathrm{Im}(\lambda_{[1]})\frac{\xi_{[1]}}{|\xi_{[1]}|^2 + 1}. \tag{5.27}$$

Similarly, the action of the matrix Darboux transformation $\mathcal{D}_{[2]}$ on q is given by

$$q_{(2)} = q - 4\,\mathrm{Im}(\lambda_{[2]})\frac{\xi_{[2]}}{|\xi_{[2]}|^2 + 1}. \tag{5.28}$$

Furthermore, the structure (5.7) of the compound Darboux transform $\Phi_{(12)} = \Phi_{(21)}$ implies that the corresponding solution $q_{(12)} = q_{(21)}$ of the nonlinear Schrödinger equation is of the form

$$q_{(12)} = G[q, \xi_{[1]}, \xi_{[2]}, \bar{\xi}_{(1)}, \bar{\xi}_{(2)}]. \tag{5.29}$$

On solving the relations (5.27), (5.28) for $\xi_{[1]}, \xi_{[2]}$, we obtain an explicit but rather involved 'nonlinear superposition principle' of the form

$$q_{(12)} = F[q, q_{(1)}, q_{(2)}, \bar{q}, \bar{q}_{(1)}, \bar{q}_{(2)}] \tag{5.30}$$

which may be used to generate iteratively solutions of the nonlinear Schrödinger hierarchy of arbitrary complexity (see, e.g., [7]).

It turns out that, remarkably, theorem 4 provides a novel compact variant of the superposition principle (5.30). This simplification is achieved by making use of the potential p defined by

$$p_x = |q|^2, \qquad p_t = \mathrm{i}(q_x\bar{q} - q\bar{q}_x) \tag{5.31}$$

corresponding to the simplest conservation law

$$[|q|^2]_t = [\mathrm{i}(q_x\bar{q} - q\bar{q}_x)]_x \tag{5.32}$$

associated with the nonlinear Schrödinger equation. It is readily verified that, up to an arbitrary constant of integration, the potential $p_{(1)}$ corresponding to the solution $q_{(1)}$ is given by

$$p_{(1)} = p - 4\,\mathrm{Im}(\lambda_{[1]})\frac{|\xi_{[1]}|^2 - 1}{|\xi_{[1]}|^2 + 1}. \tag{5.33}$$

If we make the same choice of constant of integration for the Darboux transforms $p_{(2)}, p_{(12)}$ and $p_{(21)}$ then it turns out that the permutability theorem (5.8) extends to the potential p, that is

$$p_{(12)} = p_{(21)}. \tag{5.34}$$

In this connection, it is noted that the potential p has been employed in [27] to obtain a nonlinear superposition principle containing derivatives for the 'potential nonlinear Schrödinger equation'.

'Inversion' of the relation (5.25) now yields

$$u = \tfrac{1}{2}(Q_{(1)} - Q)\mathbb{k} + \gamma\mathbb{k}, \tag{5.35}$$

where the coefficient γ is undetermined. However, since γ constitutes the $\mathbb{k}$-component of u, decomposition of $(5.25)_3$ reveals that

$$\gamma = \mathrm{Im}(\lambda_{[1]})\frac{|\xi_{[1]}|^2 - 1}{|\xi_{[1]}|^2 + 1}. \tag{5.36}$$

Accordingly, u adopts the form

$$u = -\tfrac{1}{2}\,\mathrm{Re}(q_{(1)} - q)\mathbb{i} + \tfrac{1}{2}\,\mathrm{Im}(q_{(1)} - q)\mathbb{j} - \tfrac{1}{4}(p_{(1)} - p)\mathbb{k} \tag{5.37}$$

which is precisely of the type $(5.20)_1$. It is evident that analogous expressions are valid for v, $u_{(2)}$ and $v_{(1)}$. In the current context, theorem 4 may therefore be formulated in the following manner:

Theorem 5. *Let q be a solution of the nonlinear Schrödinger equation*

$$\mathrm{i}q_t + q_{xx} + \tfrac{1}{2}|q|^2 q = 0, \tag{5.38}$$

p be a corresponding potential defined by

$$p_x = |q|^2, \qquad p_t = \mathrm{i}(q_x\bar{q} - q\bar{q}_x) \tag{5.39}$$

and $\boldsymbol{r}$ be the vector

$$\boldsymbol{r} = \begin{pmatrix} -\frac{1}{2}\,\mathrm{Re}(q) \\ \frac{1}{2}\,\mathrm{Im}(q) \\ -\frac{1}{4}p \end{pmatrix}. \tag{5.40}$$

Let the pairs $q_{(1)}, p_{(1)}$ and $q_{(2)}, p_{(2)}$ be the Darboux transforms of q, p generated by $\mathcal{D}_{[1]}$ and $\mathcal{D}_{[2]}$ associated with the Bäcklund parameters $\lambda_{[1]}$ and $\lambda_{[2]}$, respectively. If $\boldsymbol{r}_{(1)}$ and $\boldsymbol{r}_{(2)}$ are defined as in (5.40) and the unique solution $\boldsymbol{r}_{(12)}$ of the linear equation

$$\boldsymbol{r}_{(12)} - \boldsymbol{r}_{(1)} - \boldsymbol{r}_{(2)} + \boldsymbol{r} = \frac{(\boldsymbol{r}_{(12)} - \boldsymbol{r}) \times (\boldsymbol{r}_{(2)} - \boldsymbol{r}_{(1)})}{\mathrm{Re}(\lambda_{[2]}) - \mathrm{Re}(\lambda_{[1]})} \tag{5.41}$$

is parametrized according to

$$\boldsymbol{r}_{(12)} = \begin{pmatrix} -\frac{1}{2}\mathrm{Re}(q_{(12)}) \\ \frac{1}{2}\mathrm{Im}(q_{(12)}) \\ -\frac{1}{4}p_{(12)} \end{pmatrix} \tag{5.42}$$

then $q_{(12)}$ constitutes another solution of the nonlinear Schrödinger equation with $p_{(12)}$ being a corresponding potential. The pair $q_{(12)}, p_{(12)}$ is the image of both $q_{(1)}, p_{(1)}$ and $q_{(2)}, p_{(2)}$ under the Darboux transformations $\mathcal{D}^*_{[2]}$ and $\mathcal{D}^*_{[1]}$, respectively.

In conclusion, it is observed that the compact permutability theorem [28] associated with the Darboux transformation for the Calapso equation governing classical isothermic surfaces likewise originates in the inclusion of two potentials which are, in fact, intimately related to a potential which appears in Calapso's original work [29].

5.3. Generation of generalized discrete Lund–Regge surfaces

At the level of the vector-valued function $\boldsymbol{r}(x)$ employed in theorem 4, the 'closure' of the Bianchi diagram (cf figure 3) guarantees that iterative application of two families of Darboux transformations $\mathcal{D}_{[1]}^{(n_1)}$ and $\mathcal{D}_{[2]}^{(n_2)}$ corresponding to two families

$$\lambda_{[1]}(n_1), \qquad \lambda_{[2]}(n_2) \tag{5.43}$$

of Bäcklund parameters labelled by $n_1, n_2 \in \mathbb{Z}$ produces a two-dimensional 'lattice' of vector-valued functions

$$\boldsymbol{r} = \boldsymbol{r}(x; n_1, n_2). \tag{5.44}$$

For any fixed choice of the continuous variable x, the vectors $\boldsymbol{r}(n_1, n_2)$ may be regarded as representing the vertices of a quadrilateral lattice Σ of $\mathbb{Z}^2$ combinatorics. By construction, the quadrilaterals $[\boldsymbol{r}, \boldsymbol{r}_{(1)}, \boldsymbol{r}_{(12)}, \boldsymbol{r}_{(2)}]$ of Σ obey the lattice equation

$$\begin{gathered}\boldsymbol{r}_{(12)} - \boldsymbol{r}_{(1)} - \boldsymbol{r}_{(2)} + \boldsymbol{r} = \frac{(\boldsymbol{r}_{(12)} - \boldsymbol{r}) \times (\boldsymbol{r}_{(2)} - \boldsymbol{r}_{(1)})}{a(n_1) - b(n_2)} \\ a(n_1) = -\mathrm{Re}[\lambda_{[1]}(n_1)], \qquad b(n_2) = -\mathrm{Re}[\lambda_{[2]}(n_2)]\end{gathered} \tag{5.45}$$

in which the subscripts are interpreted in the usual manner as increments of the discrete variables n_1 and n_2. Accordingly, the lattice Σ constitutes a generalized discrete Lund–Regge surface with the discrete Chebyshev net relations

$$(\boldsymbol{r}_{(1)} - \boldsymbol{r})^2 = \{\mathrm{Im}[\lambda_{[1]}(n_1)]\}^2, \qquad (\boldsymbol{r}_{(2)} - \boldsymbol{r})^2 = \{\mathrm{Im}[\lambda_{[2]}(n_2)]\}^2. \tag{5.46}$$

Thus, in particular, any integrable system with an underlying $su(2)$ linear representation gives rise to generalized discrete Lund–Regge surfaces via iterative application of a matrix Darboux transformation which admits a commutative Bianchi diagram.

5.4. An alternative geometric interpretation: the Sym–Tafel formula

We now demonstrate that a four-point relation of generalized Lund–Regge type also arises in connection with the action of matrix Darboux transformations on curves and surfaces. To this end, we return to a linear equation of the form (5.1). If we adopt the definition

$$\mathsf{r} = \Phi^{-1}\Phi_\lambda \tag{5.47}$$

then the trace-free part of the matrix r evaluated at some fixed λ and denoted by

$$\tilde{r} = \mathsf{r} - \tfrac{1}{2}(\mathrm{tr}\,\mathsf{r})\mathbb{1} \tag{5.48}$$

is an element of $su(2)$. Accordingly, the usual decomposition $\tilde{r} = \tilde{\boldsymbol{r}} \cdot \boldsymbol{e}$ gives rise to a curve $\tilde{\boldsymbol{r}}(x)$ in $\mathbb{R}^3$. If (5.1) is supplemented by a compatible 'time evolution' so that $\Phi = \Phi(x, t; \lambda)$ then $\tilde{\boldsymbol{r}}(x, t)$ defines a surface in $\mathbb{R}^3$. For instance, the Lax pair (5.10), (5.11) associated with the nonlinear Schrödinger equation (5.12) gives rise to the $su(2)$ versions of the tangent vectors

$$\tilde{r}_x = \frac{1}{2}\Phi^{-1}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\Phi, \qquad \tilde{r}_t = \frac{1}{2}\Phi^{-1}\begin{pmatrix} 0 & -q \\ \bar{q} & 0 \end{pmatrix}\Phi \tag{5.49}$$

at $\lambda = 0$. Further differentiation then reveals that

$$\tilde{r}_t = \tfrac{1}{2}[\tilde{r}_x, \tilde{r}_{xx}] \tag{5.50}$$

and hence we retrieve the well-known fact that the position vector of the surfaces associated with the nonlinear Schrödinger equation is governed by the potential Heisenberg spin equation

$$\tilde{\boldsymbol{r}}_t = \tilde{\boldsymbol{r}}_x \times \tilde{\boldsymbol{r}}_{xx}. \tag{5.51}$$

The above method of relating integrable systems to surfaces has been proposed by Sym [6]. The definition (5.47) has come to be known as the 'Sym–Tafel formula' and has been used extensively in the geometric treatment of both continuous and discrete soliton equations (see, e.g., [7, 12] and references therein).

As in the preceding, we consider four matrix Darboux transformations $\mathcal{D}_{[1]}$, $\mathcal{D}_{[2]}$, $\mathcal{D}^*_{[1]}$ and $\mathcal{D}^*_{[2]}$ which admit the permutability theorem (5.8) and introduce the notation (cf (5.13))

$$\begin{aligned}\Phi_{(1)} &= P_{[1]}\mathcal{U}(\lambda)\Phi = P_{[1]}(\lambda\mathbb{1} + U)\Phi\\ \Phi_{(2)} &= P_{[2]}\mathcal{V}(\lambda)\Phi = P_{[2]}(\lambda\mathbb{1} + V)\Phi\end{aligned} \tag{5.52}$$

and

$$\begin{aligned}\Phi_{(21)} &= P^*_{[1]}P_{[2]}\mathcal{U}_{(2)}(\lambda)P^{-1}_{[2]}\Phi_{(2)} = P^*_{[1]}P_{[2]}(\lambda\mathbb{1} + U_{(2)})P^{-1}_{[2]}\Phi_{(2)}\\ \Phi_{(12)} &= P^*_{[2]}P_{[1]}\mathcal{V}_{(1)}(\lambda)P^{-1}_{[1]}\Phi_{(1)} = P^*_{[2]}P_{[1]}(\lambda\mathbb{1} + V_{(1)})P^{-1}_{[1]}\Phi_{(1)}\end{aligned} \tag{5.53}$$

for the Darboux transforms (5.2), (5.4) and (5.7). It is noted that the identity $\Phi_{(12)} = \Phi_{(21)}$ translates into

$$\mathcal{U}_{(2)}\mathcal{V} = \mathcal{V}_{(1)}\mathcal{U}. \tag{5.54}$$

Application of the Sym–Tafel formula (5.47) then yields

$$\begin{aligned}\mathsf{r}_{(1)} &= \mathsf{r} + \Phi^{-1}\mathcal{U}^{-1}\Phi, \qquad & \mathsf{r}_{(21)} &= \mathsf{r}_{(2)} + \Phi^{-1}\mathcal{V}^{-1}\mathcal{U}^{-1}_{(2)}\mathcal{V}\Phi\\ \mathsf{r}_{(2)} &= \mathsf{r} + \Phi^{-1}\mathcal{V}^{-1}\Phi, \qquad & \mathsf{r}_{(12)} &= \mathsf{r}_{(1)} + \Phi^{-1}\mathcal{U}^{-1}\mathcal{V}^{-1}_{(1)}\mathcal{U}\Phi,\end{aligned} \tag{5.55}$$

leading to the identity

$$(\mathsf{r}_{(12)} - \mathsf{r}_{(2)})(\mathsf{r}_{(2)} - \mathsf{r}) = (\mathsf{r}_{(12)} - \mathsf{r}_{(1)})(\mathsf{r}_{(1)} - \mathsf{r}) \tag{5.56}$$

by virtue of the compatibility condition (5.54) and $\mathsf{r}_{(12)} = \mathsf{r}_{(21)}$.

Decomposition of U, V and $U_{(2)}$, $V_{(1)}$ according to (5.17) now shows that

$$\begin{aligned}\mathcal{U}^{-1} &= \frac{(\lambda + a)\mathbb{1} - u}{|\lambda - \lambda_{[1]}|^2}, \qquad & \mathcal{U}^{-1}_{(2)} &= \frac{(\lambda + a)\mathbb{1} - u_{(2)}}{|\lambda - \lambda_{[1]}|^2}\\ \mathcal{V}^{-1} &= \frac{(\lambda + b)\mathbb{1} - v}{|\lambda - \lambda_{[2]}|^2}, \qquad & \mathcal{V}^{-1}_{(1)} &= \frac{(\lambda + b)\mathbb{1} - v_{(1)}}{|\lambda - \lambda_{[2]}|^2}\end{aligned} \tag{5.57}$$

so that

$$\begin{aligned}\mathsf{r}_{(1)} - \mathsf{r} &= \frac{\lambda + a}{|\lambda - \lambda_{[1]}|^2}\mathbb{1} + \tilde{r}_{(1)} - \tilde{r}\\ \mathsf{r}_{(2)} - \mathsf{r} &= \frac{\lambda + b}{|\lambda - \lambda_{[2]}|^2}\mathbb{1} + \tilde{r}_{(2)} - \tilde{r}\\ \mathsf{r}_{(12)} - \mathsf{r}_{(2)} &= \frac{\lambda + a}{|\lambda - \lambda_{[1]}|^2}\mathbb{1} + \tilde{r}_{(12)} - \tilde{r}_{(2)}\\ \mathsf{r}_{(12)} - \mathsf{r}_{(1)} &= \frac{\lambda + b}{|\lambda - \lambda_{[2]}|^2}\mathbb{1} + \tilde{r}_{(12)} - \tilde{r}_{(1)}\end{aligned} \tag{5.58}$$

and

$$\begin{aligned}(\tilde{\boldsymbol{r}}_{(1)} - \tilde{\boldsymbol{r}})^2 &= (\tilde{\boldsymbol{r}}_{(12)} - \tilde{\boldsymbol{r}}_{(2)})^2 = \frac{[\mathrm{Im}(\lambda_{[1]})]^2}{|\lambda - \lambda_{[1]}|^4}\\ (\tilde{\boldsymbol{r}}_{(2)} - \tilde{\boldsymbol{r}})^2 &= (\tilde{\boldsymbol{r}}_{(12)} - \tilde{\boldsymbol{r}}_{(1)})^2 = \frac{[\mathrm{Im}(\lambda_{[2]})]^2}{|\lambda - \lambda_{[2]}|^4}.\end{aligned} \tag{5.59}$$

The latter relations encapsulate two well-known facts [6]. Firstly, the distance between corresponding points on a curve (or surface) defined by the Sym–Tafel formula and its Darboux transform is constant along the curve (or surface). This property is referred to as the 'constant length property'. Secondly, any quadrilateral formed by a point $\boldsymbol{r}$ on a curve (or surface) and its three Darboux transforms $\boldsymbol{r}_{(1)}, \boldsymbol{r}_{(2)}$ and $\boldsymbol{r}_{(12)}$ constitutes a skew parallelogram. Accordingly, iteration of the matrix Darboux transformation as discussed in the preceding generates a discrete Chebyshev net for any fixed point on the curve (or surface).

It turns out that the discrete Chebyshev nets obtained in the above-mentioned manner are not arbitrary but are necessarily of generalized discrete Lund–Regge type. Indeed, insertion of the decompositions (5.58) into the identity (5.56) produces

$$(\tilde{a} - \tilde{b})(\tilde{r}_{(12)} - \tilde{r}_{(1)} - \tilde{r}_{(2)} - \tilde{r}) = (\tilde{r}_{(12)} - \tilde{r}_{(2)})(\tilde{r}_{(2)} - \tilde{r}) - (\tilde{r}_{(12)} - \tilde{r}_{(1)})(\tilde{r}_{(1)} - \tilde{r}), \tag{5.60}$$

where

$$\tilde{a} = \frac{\lambda + a}{|\lambda - \lambda_{[1]}|^2}, \qquad \tilde{b} = \frac{\lambda + b}{|\lambda - \lambda_{[2]}|^2}. \tag{5.61}$$

Thus, the following theorem obtains

Theorem 6. *The position vector $\tilde{\boldsymbol{r}}$ defined by the Sym–Tafel formula (5.47) and its Darboux transforms $\tilde{\boldsymbol{r}}_{(1)}, \tilde{\boldsymbol{r}}_{(2)}$ and $\tilde{\boldsymbol{r}}_{(12)}$ encoded in an analogous manner in (5.2), (5.4) and (5.7) obey the nonlinear superposition principle*

$$\tilde{\boldsymbol{r}}_{(12)} - \tilde{\boldsymbol{r}}_{(1)} - \tilde{\boldsymbol{r}}_{(2)} + \tilde{\boldsymbol{r}} = \frac{(\tilde{\boldsymbol{r}}_{(12)} - \tilde{\boldsymbol{r}}) \times (\tilde{\boldsymbol{r}}_{(2)} - \tilde{\boldsymbol{r}}_{(1)})}{\tilde{a} - \tilde{b}}, \tag{5.62}$$

where

$$\tilde{a} = \frac{\lambda - \mathrm{Re}(\lambda_{[1]})}{|\lambda - \lambda_{[1]}|^2}, \qquad \tilde{b} = \frac{\lambda - \mathrm{Re}(\lambda_{[2]})}{|\lambda - \lambda_{[2]}|^2}. \tag{5.63}$$

An illustrative consequence of the above theorem is the following nonlinear superposition principle for the potential Heisenberg spin equation:

Corollary 2. *Let Φ be a solution of the Lax pair (5.10), (5.11) and $\tilde{\boldsymbol{r}}$ be the corresponding solution of the potential Heisenberg spin equation*

$$\tilde{\boldsymbol{r}}_t = \tilde{\boldsymbol{r}}_x \times \tilde{\boldsymbol{r}}_{xx} \tag{5.64}$$

defined by the Sym–Tafel formula (5.47)$|_{\lambda=0}$. Let $\tilde{\boldsymbol{r}}_{(1)}$ and $\tilde{\boldsymbol{r}}_{(2)}$ be its Darboux transforms associated with the Bäcklund parameters $\lambda_{[1]}$ and $\lambda_{[2]}$. Then, the unique solution $\tilde{\boldsymbol{r}}_{(12)}$ of the linear equation

$$\tilde{\boldsymbol{r}}_{(12)} - \tilde{\boldsymbol{r}}_{(1)} - \tilde{\boldsymbol{r}}_{(2)} + \tilde{\boldsymbol{r}} = \frac{(\tilde{\boldsymbol{r}}_{(12)} - \tilde{\boldsymbol{r}}) \times (\tilde{\boldsymbol{r}}_{(2)} - \tilde{\boldsymbol{r}}_{(1)})}{\mathrm{Re}(\lambda_{[2]})/|\lambda_{[2]}|^2 - \mathrm{Re}(\lambda_{[1]})/|\lambda_{[1]}|^2} \tag{5.65}$$

is another solution of the potential Heisenberg spin equation and constitutes Darboux transforms of both $\tilde{\boldsymbol{r}}_{(1)}$ and $\tilde{\boldsymbol{r}}_{(2)}$.

The above analysis implies that (5.55) provides a link between solutions of the two four-point relations of generalized Lund–Regge type contained in theorems 4 and 6. Accordingly, iteration of the matrix Darboux transformation gives rise to a connection between solutions of the two associated discrete generalized Lund–Regge equations and corresponding discrete surfaces. Indeed, without reference to the genesis of this connection, this fact may be formulated in the following manner:

Theorem 7. *Let $\boldsymbol{r}$ be a solution of the generalized discrete Lund–Regge equation*

$$\Delta_{12}\boldsymbol{r} = \frac{(\boldsymbol{r}_{12} - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{a(n_1) - b(n_2)} \tag{5.66}$$

and Φ be an 'eigenfunction' obeying the linear pair of difference equations

$$\Phi_1 = [(\lambda + a)\mathbb{1} + \Delta_1 r]\Phi, \qquad \Phi_2 = [(\lambda + b)\mathbb{1} + \Delta_2 r]\Phi \tag{5.67}$$

with $r = \boldsymbol{r}\cdot\boldsymbol{e}$. Then, the pair

$$\Delta_1\tilde{r} = -\frac{\Phi^{-1}\Delta_1 r\Phi}{(\lambda+a)^2 + (\Delta_1\boldsymbol{r})^2}, \qquad \Delta_2\tilde{r} = -\frac{\Phi^{-1}\Delta_2 r\Phi}{(\lambda+b)^2 + (\Delta_2\boldsymbol{r})^2} \tag{5.68}$$

is compatible and $\tilde{\boldsymbol{r}}$ defined by $\tilde{r} = \tilde{\boldsymbol{r}}\cdot\boldsymbol{e}$ constitutes another solution of (5.66) with

$$\tilde{a} = \frac{\lambda+a}{(\lambda+a)^2 + (\Delta_1\boldsymbol{r})^2}, \qquad \tilde{b} = \frac{\lambda+b}{(\lambda+b)^2 + (\Delta_2\boldsymbol{r})^2}. \tag{5.69}$$

The corresponding first integrals are given by

$$(\Delta_1\tilde{\boldsymbol{r}})^2 = \frac{(\Delta_1\boldsymbol{r})^2}{[(\lambda+a)^2 + (\Delta_1\boldsymbol{r})^2]^2}, \qquad (\Delta_2\tilde{\boldsymbol{r}})^2 = \frac{(\Delta_2\boldsymbol{r})^2}{[(\lambda+b)^2 + (\Delta_2\boldsymbol{r})^2]^2}. \tag{5.70}$$

It is evident that, in general, the transformation $\boldsymbol{r} \to \tilde{\boldsymbol{r}}$ does *not* preserve a, b and the first integrals $(\Delta_1\boldsymbol{r})^2, (\Delta_2\boldsymbol{r})^2$. This property distinguishes the above transformation from the matrix Darboux transformation set down in theorem 2. However, if we choose $\lambda = 0$ then the constraints

$$a^2 + (\Delta_1\boldsymbol{r})^2 = 1, \qquad b^2 + (\Delta_2\boldsymbol{r})^2 = 1 \tag{5.71}$$

associated with the discrete Lund–Regge equation (3.22) and the discrete $O(4)$ nonlinear σ-model (3.11) guarantee that the quantities a, b and $(\Delta_1\boldsymbol{r})^2, (\Delta_2\boldsymbol{r})^2$ remain the same. This underlines the privileged nature of the discretization (3.22). In fact, under the assumptions (5.71) and $\lambda = 0$, the pair (5.68) may be brought into the form

$$\Phi_1^\dagger = (a\mathbb{1} + \Delta_1\tilde{r})\Phi^\dagger, \qquad \Phi_2^\dagger = (b\mathbb{1} + \Delta_2\tilde{r})\Phi^\dagger, \tag{5.72}$$

which shows that $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ represent the position vectors of the pairs of discrete Lund–Regge surfaces Σ and $\tilde{\Sigma}$ introduced in section 3.3 with $N = \Phi$ and $N^\dagger = \Phi^\dagger$ being the solutions of the discrete $SU(2)$ chiral models (3.32) and (3.33), respectively.

On application of the scaling (4.41), the above theorem reduces in the formal continuum limit to the following:

Theorem 8. *Let $\boldsymbol{r}$ be a solution of the generalized Lund–Regge equation*

$$\boldsymbol{r}_{xy} = 2\frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{a(x) - b(y)} \tag{5.73}$$

and Φ be an 'eigenfunction' obeying the linear pair of difference equations

$$\Phi_x = \frac{r_x}{\lambda+a}\Phi, \qquad \Phi_y = \frac{r_y}{\lambda+b}\Phi \tag{5.74}$$

with $r = \boldsymbol{r}\cdot\boldsymbol{e}$. Then, the pair

$$\tilde{r}_x = -\frac{\Phi^{-1}r_x\Phi}{(\lambda+a)^2}, \qquad \tilde{r}_y = -\frac{\Phi^{-1}r_y\Phi}{(\lambda+b)^2} \tag{5.75}$$

is compatible and $\tilde{\boldsymbol{r}}$ defined by $\tilde{r} = \tilde{\boldsymbol{r}}\cdot\boldsymbol{e}$ constitutes another solution of (5.66) with

$$\tilde{a} = \frac{1}{\lambda+a}, \qquad \tilde{b} = \frac{1}{\lambda+b}. \tag{5.76}$$

The corresponding first integrals are given by

$$\tilde{\boldsymbol{r}}_x^2 = \frac{\boldsymbol{r}_x^2}{(\lambda+a)^4}, \qquad \tilde{\boldsymbol{r}}_y^2 = \frac{\boldsymbol{r}_y^2}{(\lambda+b)^4}. \tag{5.77}$$

Once again, in the case of the Lund–Regge equation (2.25) corresponding to $a = -b = 1$, the transformation $\boldsymbol{r} \to \tilde{\boldsymbol{r}}$ with $\lambda = 0$ represents the link between the two Lund–Regge surfaces Σ and $\tilde{\Sigma}$ encapsulated in the pair (2.30). The (geometric) nature of the generic transformation for the generalized (discrete) Lund–Regge equation as formulated in theorems 7 and 8 is currently under investigation.

References

[1] Lund F and Regge T 1976 Unified approach to strings and vortices with soliton solutions *Phys. Rev.* D **14** 1524–35

[2] Ablowitz M J, Kaup D J, Newell A C and Segur H 1973 Nonlinear evolution equations of physical significance *Phys. Rev. Lett.* **31** 125–7

[3] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (*SIAM Studies in Applied Mathematics* vol 4) (Philadelphia: SIAM)

[4] Pohlmeyer K 1976 Integrable Hamiltonian systems and interactions through quadratic constraints *Commun. Math. Phys.* **46** 207–21

[5] Lund F 1977 Note on the geometry of the nonlinear σ model in two dimensions *Phys. Rev.* D **15** 1540–3

[6] Sym A 1985 Soliton surfaces and their applications *Geometric Aspects of the Einstein Equations and Integrable Systems* ed R Martini (Berlin: Springer)

[7] Rogers C and Schief W K 2002 *Darboux and Bäcklund Transformations. Geometry and Modern Applications in Soliton Theory* (*Cambridge Texts in Applied Mathematics* (Cambridge: Cambridge University Press)

[8] Bobenko A and Pinkall U 1996 Discrete surfaces with constant negative Gaussian curvature and the Hirota equation *J. Diff. Geom.* **43** 527–611

[9] Sauer R 1950 Parallelogrammgitter als Modelle pseudosphärischer Flächen *Math. Z.* **52** 611–22

[10] Wunderlich W 1951 Zur Differenzengeometrie der Flächen konstanter negativer Krümmung *Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. II* **160** 39–77

[11] Hirota R 1977 Nonlinear partial difference equations: III. Discrete sine-Gordon equation *J. Phys. Soc. Japan* **43** 2079–86

[12] Bobenko A I and Seiler R (ed) 1999 *Discrete Integrable Geometry and Physics* (Oxford: Clarendon)

[13] do Carmo M P 1976 *Differential Geometry of Curves of Surfaces* (Englewood Cliffs, NJ: Prentice-Hall)

[14] Chebyshev P L 1955 On the cutting of our clothes *Collected Works* vol 5 (Moscow: Fizmatgiz) pp 165–70

[15] Eisenhart L P 1960 *A Treatise on the Differential Geometry of Curves and Surfaces* (New York: Dover)

[16] Nappi C R 1980 Some properties of an analog of the chiral model *Phys. Rev.* D **21** 418–20

[17] Burtsev S P, Zakharov V E and Mikhailov A V 1987 Inverse scattering method with variable spectral parameter *Teoret. Mat. Fiz.* **70** 323–41

[18] Kramer D 1982 Equivalence of various pseudopotential approaches for Einstein–Maxwell fields *J. Phys. A: Math. Gen.* **15** 2201–7

[19] Hoffmann T 2000 On the equivalence of the discrete nonlinear Schrödinger equation and the discrete isotropic Heisenberg magnet *Phys. Lett.* A **265** 62–7

[20] Nijhoff F W and Walker A J 2001 The discrete and continuous Painlevé hierarchy and the Garnier system *Glasgow Math. J.* **43A** 109–23

[21] Bobenko A I and Suris Yu B 2002 Integrable systems on quad-graphs *Int. Math. Res. Not.* **2002** 573–611

[22] Nijhoff F W 2002 Lax pair for the Adler (lattice Krichever–Novikov) system *Phys. Lett.* A **297** 49–58

[23] Adler V E, Bobenko A I and Suris Yu B 2003 Classification of integrable equations on quad-graphs. The consistency approach *Commun. Math. Phys.* **233** 513–43

[24] Zakharov V E and Manakov S V 1985 Construction of multidimensional nonlinear integrable systems and their solutions *Funct. Anal. Pril.* **19** 11–25

[25] Darboux G 1910 *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes* (Paris: Gauthier-Villars)

[26] Bogdanov L V and Konopelchenko B G 1995 Lattice and q-difference Darboux–Zakharov–Manakov systems via $\bar{\partial}$ method *J. Phys. A: Math. Gen.* **28** L173–8

[27] Boiti M, Laddomada C and Pempinelli F 1981 An equivalent real form of the nonlinear Schrödinger equation and the permutability for Bäcklund transformations *Nuovo Cimento* B **62** 315–26

[28] Schief W K 2001 Isothermic surfaces in spaces of arbitrary dimension: integrability, discretization and Bäcklund transformations. A discrete Calapso equation *Stud. Appl. Math.* **106** 85–137

[29] Calapso P 1903 Sulla superficie a linee di curvatura isoterme *Rend. Circ. Mat. Palermo* **17** 275–86